What is the Real Role of a Division Operation in a Computer-Aided Geometric Design System ?

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Abstract

Introduction

It is of vital importance that a computer-aided geometric design system is sufficiently reliable. The present geometric processing, which we call Euclidean Processing, has, however, inherent problems of "inaccuracy", "instability" and "complexity" which have a direct bearing on a system's reliability because a division operation is the root cause of all such problems according to the present author's course of study[1] and the operation is inevitable in the Euclidean Processing. Therefore there will be a limit to the level of reliability achievable in a system based on the Euclidean Processing.

This observation leads us necessarily to use homogeneous coordinates instead of Euclidean coordinates because the former coordinates are considered to be pre-division data immediately before computing Euclidean coordinates by division operations while the latter post-division data. Throughout the present article, the expression "Homogeneous Processing", which employs pre-division, homogeneous coordinates, is used in contradistinction to "Euclidean Processing" which employs post-division, Euclidean coordinates.

For the last several years the present author has proposed Totally Four-Dimensional Homogeneous Processing or Homogeneous Processing in short in order to achieve geometric processing without using a division operation, where geometric elements are defined in either classical projective space $\mathbf{P}^3$ or two-sided space $\mathbf{T}^3$, these vector space models are processed in an extended four-dimensional vector space, the principle of duality is made the most of, and projective invariance of geometry representation and computation is pursued.

The playground for the Homogeneous Processing is not a projective space itself but the associated vector space of a projective space. Therefore the Homogeneous Processing is essentially different from the scheme proposed by Stolfi [2] or described by Farin [3].

A Paradox

We have compared the characteristics of the Homogeneous Processing with those of the Euclidean Processing in terms of many evaluation items and found that the result turned out to be surprisingly favorable for the Homogeneous Processing and appeared to be contradictory to our common sense because the former is superior to the latter in almost all items. The result claims a paradox holds such that

> the Homogeneous Processing which provides the most general technique is, in reality, the most simple, stable, and accurate, and furthermore provides the properties of generality, uniformity and duality which are really desirable in constructing a systematically organized computer-aided geometric design system.

Then why does such a paradox hold in the Homogeneous Processing ? The reason of the paradox would have to be explained clearly.

Why does the paradox hold ?

We will take up the pentagon problem. Please, refer to pp.122-124 of [4] for the problem. In the problem, the output of the intersection computation of two line segments becomes the input to the next. The intersection computation can be written as follows:

$$\begin{aligned} a_0x + b_0y + c_0 &= 0, \\ a_1x + b_1y + c_1 &= 0, \end{aligned}$$

$$\Delta_x = \begin{vmatrix} -c_0 & b_0 \\ -c_1 & b_1 \end{vmatrix},\ \Delta_y = \begin{vmatrix} a_0 & -c_0 \\ a_1 & -c_1 \end{vmatrix},\ \Delta = \begin{vmatrix} a_0 & b_0 \\ a_1 & b_1 \end{vmatrix}, \tag{1}$$

$$x = \frac{\Delta_x}{\Delta},\ y = \frac{\Delta_y}{\Delta}. \tag{2}$$

In the process above, a small amount of error is unavoidable in the division operation of eq.(2). There, the (x, y) is derived, which includes the error and is input to the next process. Therefore the error is accumulated by the repetition of the process. Here we must note that the three dimensional homogeneous coordinates $(\Delta_x, \Delta_y, \Delta)$ of the intersection point have already been derived ! This example is the case of plane geometry. In the case of space geometry we can say the four dimensional homogeneous coordinates of the intersection point have already been derived ! Furthermore we must note that the homogeneous coordinates $(\Delta_x, \Delta_y, \Delta)$ can be represented exactly in long integers. There is no need of deriving Euclidean coordinates (x, y) by carrying out harmful division operations. Our experiment demonstrates complete restoration of the homogeneous coordinates of the original input vertex data after iterating going-in operation m times and then going-out operation m times in the Homogeneous Processing. In this experiment we performed a normalization of integer homogeneous coordinates after each going-in or -out operation, where the normalization means the reduction of the greatest common divisor among the integer coordinates.

As we see in the example above, the Euclidean Processing can be considered to create four dimensional(space geometry case) homogeneous coordinates in the beginning. Nevertheless it converts the homogeneous coordinates to Euclidean ones by performing harmful division operations. Here there is no essential reason for doing such division operations. This is really unnatural processing ! It would not be difficult to justify the above mentioned paradox if we consider that the Euclidean Processing does such unnecessary and harmful division operations.

The observation above would be difficult to be derived from the conventional Euclidean point of view and only be possible from a higher dimensional point of view.

The Real Role of an Arithmetic Division Operation

As long as next processing continues further, there is no need of carrying out any division operation. The homogeneous coordinate or coefficient data are sufficient for subsequent processing. Division operations during each geometric processing do not create any benefit but harmful influences.

Then what is the real role of an arithmetic division operation in a computer-aided geometric design system ? It lies in the very end of the whole process,where the result is to be interfaced to the designer for the confirmation of his design intention. In this case homogeneous data must be converted to Euclidean data by division operations in order that he can easily understand the result. Ordinarily the division operations are carried out in performing perspective projections.

Conclusion

We would like to claim that we had better shift the playground for geometric processing from Euclidean to homogeneous one. We are organizing a research committee with Japanese industry engineers and examining the possibility of putting the Totally Four-dimensional Geometric Processing approach into practical use.

References

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[4] C. M. Hoffmann: Geometric and Solid Modeling —-An Introduction, pp.122-124, Morgan Kaufmann Publishers,Inc., California, (1989).