Optimal Compaction of Orthogonal Representations*

(Abstract)

Stina Bridgeman† Giuseppe Di Battista‡ Walter Didimo‡ Giuseppe Liotta§

Roberto Tamassia† Luca Vismara†

1 Introduction

Orthogonal drawings are drawings of graphs in which every edge is represented as a chain of horizontal and vertical segments. In this paper we consider the problem of producing an orthogonal grid drawing of a planar graph with a given orthogonal representation. The *orthogonal representation* captures the shape of a planar orthogonal drawing, and consists of a description of the bends along each edge (and whether each bend is a left turn or right turn) and the angles between consecutive edges around each vertex. Each bend and angle is a multiple of $\pi/2$. Orthogonal representations are related to rectilinear graphs, introduced in [6].

An orthogonal drawing is generated from the orthogonal representation by assigning integer lengths to the horizontal and vertical segments so that no two segments overlap or cross, in a process known as *compaction*. The goal of compaction is to keep the area of the final drawing small while satisfying the non-overlap constraints.

A sketch of a compaction algorithm using a rectangular decomposition of the faces of the orthogonal representation is given in [4] and described in more detail in [1]. Once the faces have been decomposed into rectangles, a reduction to a minimum cost flow problem can be used to minimize the width, height, and total edge length of the drawing. Alternatively, a critical path method can be used to minimize the width and height of the drawing [1]. The resulting drawing, however, may not be the optimal drawing given a particular orthogonal representation (Figure 1). Problems arise because in Figure 1b, v_1 is below v_2 but not left of v_2, whereas in Figure 1c, v_1 is left of v_2 but not below v_2. Note that Figure 1c has both a smaller area (8 units vs. 9) and a shorter total edge length (13 units vs. 14 units) than Figure 1b.

In this paper we present a characterization of orthogonal representations which can be optimally compacted in an efficient manner and provide an algorithm for the compaction.

2 Turn-Regularity

Let H be an orthogonal representation. Without loss of generality, H can be assumed to have no bends since any bends can be replaced by dummy vertices. Let Γ be any drawing of H. $x(v)$ and $y(v)$ are the x- and y-coordinates of vertex v, respectively.

An *orthogonal relation* between two vertices v_1 and v_2 in Γ is one of:

- v_1 is left of v_2 (written $v_1 <_x v_2$) if $x(v_1) < x(v_2)$
- v_1 is below v_2 (written $v_1 <_y v_2$) if $y(v_1) < y(v_2)$

A *compulsory orthogonal relation* between two vertices is one which holds for all drawings of H (up to rotations by a multiple of $\pi/2$).

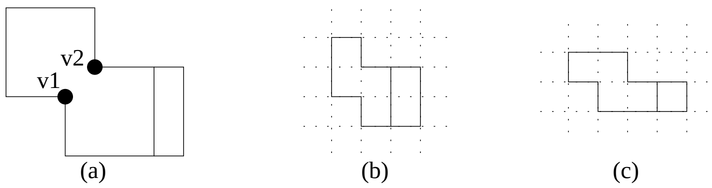

Figure 1: (a) An orthogonal representation of a planar graph G. (b) and (c) are two possible orthogonal drawings of G.

*Research supported in part by the CNR Project "Geometria Computazionale Robusta con Applicazioni alla Grafica ed al CAD", the ESPRIT LTR Project no. 20244 - ALCOM-IT, the National Science Foundation under grants CCR-9732327 and CDA-9703080, and the U.S. Army Research Office under grant DAAH04-96-1-0013.

†Center for Geometric Computing, Dept. of Comp. Sci., Brown Univ., Providence, RI 02912. `[ssb|rt|lv]@cs.brown.edu`

‡Dip. di Inf. e Aut., Università di Roma Tre, via della Vasca Navale 79, 00146 Roma, Italy. `[gdb|didimo]@dia.uniroma3.it`

§Dip. di Inf. e Sist., Università di Roma "La Sapienza", via Salaria 113, 00198 Roma, Italy. `liotta@dis.uniroma1.it`

To characterize those orthogonal representations which have a compulsory orthogonal relation for each pair of vertices, we introduce the notion of *turn-regularity.*

In the following, always traverse the boundary of a face f so that the face is on the left. For each face f of H, annotate the vertices and edges of f as follows:

- For each edge e of f, let $next(e)$ be the edge following e on the boundary of f and let $corner(e)$ be the common vertex between e and $next(e)$. For each vertex v of f, let $prev(v)$ and $next(v)$ be the edges e_p and e_v such that $corner(e_p) = v$ and $next(e_p) = e_v$.
- For each vertex $v = corner(e)$ of f, let $turn(v) = 1$ if the angle between e and $next(e)$ (the angle internal to the face) is $\pi/2$ (a left turn), $turn(v) = 0$ if the angle is π (no turn), and $turn(v) = -1$ if the angle is $3\pi/2$ (a right turn). The vertex v is a *inflex* vertex if $turn(v) = \pi/2$, a *flat* vertex if $turn(v) = \pi$, and a *reflex* vertex if $turn(v) = 3\pi/2$.
- For a pair of edges e_1 and e_2 of f, let $rotation(e_1, e_2) = \sum_v turn(v)$ for all vertices v on the boundary of f from $corner(e_1)$ (included) to $corner(e_2)$ (excluded). $rotation(e_1, e_2)$ is the *rotation* of edge e_2 with respect to edge e_1.

These definitions are based on those used in [1, 5].

A *bad pair* is a pair of reflex vertices v_1 and v_2 such that $rotation(prev(v_1), prev(v_2)) = 2$. In Figure 1, vertices v_1 and v_2 form a bad pair. A face f is *turn-regular* if there is no bad pair of vertices. H is turn-regular if all faces of H are turn-regular.

Theorem 1 *For an orthogonal representation H, H is turn-regular if and only if there is a compulsory orthogonal relation between every pair of vertices of H.*

3 Switch-Regularity

Turn-regularity is related to the idea of regularity of upward planar embedded digraphs defined in [2], called *switch-regularity* here to distinguish it from turn-regularity.

Let $\overrightarrow{\Gamma}$ be an orientation of Γ such that all vertical segments are directed upward and all horizontal segments are directed rightward, and let $\overleftarrow{\Gamma}$ be an orientation of Γ such that all vertical segments are directed upward and all horizontal segments are directed leftward. Observe that $\overrightarrow{\Gamma}$ is a planar upward drawing in the North-East direction and that $\overleftarrow{\Gamma}$ is a planar upward drawing in the North-West direction. Both $\overrightarrow{\Gamma}$ and $\overleftarrow{\Gamma}$ induce an orientation on H. We shall denote such orientations as $\overrightarrow{H}$ and $\overleftarrow{H}$, respectively.

Theorem 2 *A planar orthogonal representation H is turn-regular if and only if $\overrightarrow{H}$ and $\overleftarrow{H}$ are both switch-regular.*

4 Compaction

Lemma 1 *Recognizing a turn-regular orthogonal representation can be done in $O(n)$ time and space.*

Theorem 3 *A planar orthogonal drawing with optimal area of a turn-regular orthogonal representation with n vertices can be constructed in $O(n)$ time and space.*

Theorem 4 *A planar orthogonal drawing with optimal area and optimal total edge length of a turn-regular orthogonal representation with n vertices can be constructed in $O(n^{7/4} \log n)$ time and $O(n)$ space.*

The complexity of computing the minimum cost flow used in Theorem 4 is shown in [3].

References

[1] G. Di Battista, P. Eades, R. Tamassia, and I. G. Tollis. *Graph Drawing.* Prentice Hall, 1998.

[2] G. Di Battista and G. Liotta. Upward planarity checking: "Faces are more than polygons". In *Graph Drawing (Proc. GD '98)*, Lecture Notes Comput. Sci. Springer-Verlag.

[3] A. Garg and R. Tamassia. A new minimum cost flow algorithm with applications to graph drawing. In S. C. North, editor, *Graph Drawing (Proc. GD '96)*, Lecture Notes Comput. Sci. Springer-Verlag, 1997.

[4] R. Tamassia. On embedding a graph in the grid with the minimum number of bends. *SIAM J. Comput.*, 16(3):421–444, 1987.

[5] R. Tamassia, I. G. Tollis, and J. S. Vitter. Lower bounds and parallel algorithms for planar orthogonal grid drawings. In *Proc. IEEE Symposium on Parallel and Distributed Processing*, pages 386–393, 1991.

[6] G. Vijayan and A. Wigderson. Rectilinear graphs and their embeddings. *SIAM J. Comput.*, 14:355–372, 1985.