Finding Basis Functions for Pyramidal Finite Elements *

Vasiliki Chatzi †

September 30, 1998

Abstract

The finite-element method is used to obtain approximate solutions of partial differential equations defined on a physical domain. The method is based on discretizing this domain by dividing it into simple shapes, called *finite elements*. The resulting collection of elements is called a *finite element mesh*. The quality of the mesh affects both the efficiency and the accuracy of the method [4, 7]. Once the mesh has been created, nodal points are defined on the elements, a *basis function* is associated with each nodal point, and a system of linear equations is constructed and solved. Lagrangian elements, which are elements that have the same number of nodal points placed on each edge and internal points placed on a grid [2, 7], are commonly used today.

For many problems, meshes that consist mainly of cubic elements are considered ideal [1]. Often, the same problems also exhibit regions where the solution changes rapidly, which should be covered with smaller elements. For problems with these characteristics we are developing *crystalline meshes* (work in progress, soon to be documented) that use pyramids in order to make transitional regions conformal [3, 6]. Pyramids are a natural choice for transitional elements, but they are not widely used, because finding Lagrangian basis functions for them is quite complex. The general techniques developed for constructing basis functions [6, 7] involve solving a system of linear equations which, in the case of the pyramid, is singular. Indeed, we can prove the following theorem:

Theorem 1 *It is impossible to construct polynomial basis functions for the first-order Lagrangian pyramid.*

Notice that the above theorem implies that the approach for constructing basis functions for finite elements suggested by Hughes and Akin [5] does not work for pyramids. It is possible, however, to construct non-polynomial, well-behaved basis functions for Lagrangian pyramids. First of all, let us give a slightly modified definition of Lagrangian pyramids:

Definition 1 Lagrangian pyramids *are pyramids that have Lagrangian faces and no nodal points placed in their interior.*

By disallowing internal nodal points we retain the continuity at the common faces of two elements, while reducing the number of unknowns in the final system of equations. The following theorem allows us to focus on finding basis functions for a standard pyramid (to be used as a template), conveniently positioned in the coordinate system (similar to the theorem for tetrahedral and parallelepiped elements [6]).

Theorem 2 *Given a set of basis functions for a Lagrangian pyramid, basis functions for all other Lagrangian pyramids of the same order can be obtained by applying a linear transformation on the given functions.*

*Research supported in part by the National Science Foundation under grand CCR 9400232 and by the U.S. Army Research Office under grand DAAH04-96-1-0013.

†Department of Computer Science, Brown University, Box 1910, Providence, RI 02912. E-mail: `vc@cs.brown.edu`

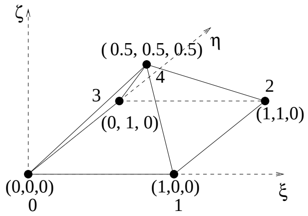

Figure 1: Template pyramid

The template pyramid we chose is shown in Figure 1. For this pyramid, we can give a set of first-order basis functions that are ratios of polynomials (shown in Table 1). Four of the functions have one singular point, coinciding with the fourth vertex of the pyramid. Nevertheless, they are well-behaved and do not present mathematical problems when constructing the matrices used in the computation.

$$\begin{aligned} \phi_0(\xi,\eta,\zeta) &= -\frac{(\xi+\zeta-1)(\eta+\zeta-1)}{2\zeta-1} & \qquad \phi_3(\xi,\eta,\zeta) &= \frac{(\eta-\zeta)(\xi+\zeta-1)}{2\zeta-1} \\ \phi_1(\xi,\eta,\zeta) &= \frac{(\xi-\zeta)(\eta+\zeta-1)}{2\zeta-1} & \qquad \phi_4(\xi,\eta,\zeta) &= 2\zeta \\ \phi_2(\xi,\eta,\zeta) &= -\frac{(\eta-\zeta)(\xi-\zeta)}{2\zeta-1} \end{aligned}$$

Table 1: Basis functions for the 5 nodal points of the first order pyramid.

These functions are the foundation of an inductive technique for the construction of basis functions for pyramids of an arbitrary order k. Using these as a base case we can give a constructive proof of the following theorem:

Theorem 3 *For any $k > 0$, there exist basis functions for the kth-order template Lagrangian pyramid, which are ratios of polynomials.*

The constructive nature of the proof makes it possible to automate the process of generating basis functions for pyramids of any order, as well as the process of constructing the linear systems of equations. Moreover, we are convinced that the same approach can be used to find basis functions for other "special" finite element shapes, which do not allow polynomial basis functions, but might be appropriate in cases where conformity is essential.

Acknowledgments: Special thanks to Paul Fisher for suggesting the problem, and to Franco Preparata for his guidance on this research.

References

[1] S.E. Benzley, E. Perry, B. Merkley, K. Clark, and G. Sjaardema. A comparison of all hexagonal and all tetrahedral finite element meshes for elastic and elastic-plastic analysis. *4th International Meshing Roundtable*, pages 179–192, 1995.

[2] David S. Burnett. *Finite elements analysis: from concepts to applications.* Addison-Wesley, 1987.

[3] G. F. Carey. *Computational grids.* Taylor & Francis, 1989.

[4] P. L. George. *Automatic mesh generation: Application to finite element methods.* John Wiley & Sons, 1991.

[5] T.J.R. Hughes and J. E. Akin. Techniques for developing "special" finite element shape functions with particular reference to singularities. *Int. J. Numer. Meth. Eng.*, 15:733–751, 1980.

[6] H. R. Schwarz and J. R Whiteman. *Finite element methods.* Academic press, 1988.

[7] O. C. Zienkiewicz and R. L. Taylor. *The finite element method.* McGraw-Hill, 1994.