# Computing Morse Functions on Triangulated Manifolds

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## 1 Introduction

In this paper we introduce an algorithm to compute a discrete Morse function on an abstract or geometrically realized triangulated manifold of arbitrary dimension. Critical points of a Morse function are related to topological properties of the manifold, so this information can be used to aid in topological analysis of the manifold.

For details about Morse theory the standard text is Milnor [7]. In classical Morse theory, a smooth function on a smooth manifold is a Morse function if it contains only nondegenerate (isolated) critical points. One of the results of Morse theory says that the $d$-dimensional manifold is homotopy equivalent to a cell complex with one cell of dimension $\lambda$ for each critical point of Morse index $\lambda$, $0 \leq \lambda \leq d$, and other results put bounds on the Betti numbers of the manifold.

For the discrete case, let $K$ be a $k$-dimensional simplicial complex, and let $K$ be embedded in some space $\mathbb{R}^n$. Define a height function $f : \mathbb{R}^n \to \mathbb{R}$ which takes points from $\mathbb{R}^n$ to a line. $f$ is *general for $K$* if $f(v) \neq f(w)$ whenever $v$ and $w$ are the vertices of an edge in $K$. If $f$ is general for $K$, then $f$ is a Morse function if $K$ is a triangulated manifold [3].

Past use of Morse theory in computation has been in the domain of smooth manifolds, and it has been assumed that a function on the manifold is included in the given input. For example, Morse theory has been suggested for applications such as surface coding and reconstruction [10], and implicit surface modeling [9]. There has been some recent work on discrete Morse theory [6], but in that case the function must be provided and it is not in any way a natural extension of the smooth case.

Our work is fundamentally different from past work in that we are computing a natural Morse function given only the triangulated domain as input, our aim being to provide a tool for structural analysis of the domain. For complexes that can be embedded in the 3-sphere, there exist efficient algorithms to compute Betti numbers [4], which are topological invariants of the complex. However, Betti numbers only provide global information about the complex; they do not give local structural information at each vertex, as does a Morse function. When searching for particular topological features in a large dataset (e.g., one representing a macromolecule), a scientist needs to know not only that these features exist, but where they are. The output of the computation can then be represented with a visual display or an auditory display.

This application extends to complexes of dimension greater than 3. In addition, we are exploring the possibilities for using Morse theory to compute Betti numbers in higher dimensions, where no efficient algorithms exist.

## 2 Results

Our method for computing a general function $f$ on $K$ proceeds as follows. For every vertex $v \in K$, we compute the distance $d(u, v)$ (minimum number of edges) from an arbitrary start vertex $u$. The distance $d(u, \sigma)$ on a simplex $\sigma$ of dimension greater than 0 is equal to the minimum distance on any of its vertices. Let $W_K(i), i \geq 0$, be the set of $\sigma \in K$ such that $d(u, v) = i$ for all vertices $v \in \sigma$, and $\sigma$ is face of some simplex $\tau \in K$ with $d(u, \tau) = i - 1$. We call each set $W_K(i)$ a *wave*, and we call the entire set $W_K$ the wave subcomplex of $K$.

Let $S_K$ be the set of simplices of $K$ that do not belong to $W_K$; $S_K(i)$ is the set of simplices in $S_K$ such that $d(u, \sigma) = i$. We would like to extend $d(u, v)$ linearly to a continuous function $d(u, x)$ over all points of $K$, so that $d(u, x) = i$ for all points of $\sigma \in W_K(i)$, and $d(u, x)$ varies from $i$ to $i+1$ for all points of $\sigma \in S_K(i)$. For most $\sigma \in S_K(i)$ a linear interpolation presents no problem, since some $v \in \sigma$ belong to $W_K(i)$ and some belong to $W_K(i + 1)$. However, it is possible to have a simplex in $S_K(i)$ which has all vertices in $W_K(i)$. We call such a simplex a *grounded simplex*, see figure 1.

The barycentric subdivision sd$K$ of $K$ (see [8] or another standard text for definition) has the same topological properties as $K$, and we can eliminate grounded simplices with barycentric subdivision. Let $W_{\mathrm{sd}K}$ be the wave subcomplex of sd$K$, and let $S_{\mathrm{sd}K}$ be defined accordingly. The start vertex $u$ is defined to be the same for both $W_K$ and $W_{\mathrm{sd}K}$. We prove that we can eliminate grounded simplices from the wave traversal

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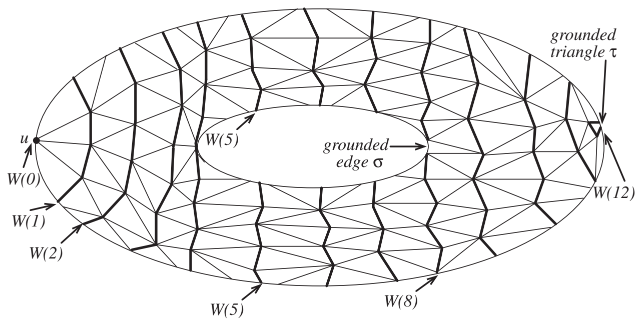


Figure 1: Waves on a 2-dimensional torus. $W(8)$ includes both vertices of edge $\sigma$ but not $\sigma$ itself. Therefore, $\sigma$ is a grounded edge in $S(8)$. $W(12)$ includes all the edges of triangle $\tau$ but not $\tau$ itself, so $\tau$ is a grounded triangle in $S(12)$.

using barycentric subdivision in both [1] and [2].

We define a continuous function $f$ using linear interpolation of the distances on vertices in sd$K$, but $f$ is not general for sd$K$, since all points $x$ in a wave $W_{\mathrm{sd}K}(i)$ have the same value $d(u,x) = i$. So, we do not yet have a Morse function, but we can produce one by perturbing $f$ slightly using repeated barycentric subdivisions and recursion.

Let $u_0$ be an original vertex of $K$. We compute $d(u_0, x)$ on the $k$th barycentric subdivision $\mathrm{sd}^{(k)}K$. Since each $W_{\mathrm{sd}^{(k)}K}(i)$ is itself a $(k-1)$-dimensional complex, wave traversal is defined on its components. We choose a vertex $u_1$ in each component of each wave $W_{\mathrm{sd}^{(k)}K}(i)$ and compute $d_1(u_1, x)$, the distance from $u_1$ to $x$. Now each wave on each component in the recursive wave traversal is itself a $(k-2)$-complex, and so we can compute waves on those, and so forth. We compute the wave traversal recursively $k$ times, so that the waves resulting from the final wave traversal are 0-complexes.

Let $d_i$ be the distance function from $u_i$ on a component of a wave defined by $d_{i-1}(u_{i-1}, x)$, where $d_0 = d$. We replace $f$ on each vertex $x$ with

$$\begin{aligned} f(x) &= \epsilon^0 d_0(u_0, u_1) - \cdots - \epsilon^{k-1} d_{k-1}(u_{k-1}, x) \\ &= \epsilon^0 d_0(u_0, x) - \cdots - \epsilon^{k-1} d_{k-1}(u_{k-1}, x) \ , \end{aligned}$$

for some sufficiently small $\epsilon > 0$. Now it is straightforward to prove that $f$ is general for $\mathrm{sd}^{(k)}K$ and therefore a Morse function.

## 3 Current and Future Work

While of theoretical interest, the technique outlined here is not practical for computing a Morse function, since computing the $k$th barycentric subdivision results in exponential growth of the size of the dataset. In [1] we describe an algorithm based on this work for computing a Morse function and Morse indices of critical points on 2-manifolds in time $O(n\alpha(n))$ and space $O(n)$ ($\alpha(n)$ is the slowly growing inverse of Ackermann's function). We are currently extending this to make the algorithm practical for higher dimensional manifolds, and we are extending the theory to general simplicial complexes using a generalization of Morse theory.

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