Folding Any Silhouette from a Strip

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A classic open question in origami mathematics is whether every simple polygon is the silhouette of a flat origami. This question was first formally stated by Bern and Hayes in SODA'96 [3]. Here we show the answer is yes.

A related problem in origami design is to take *bicolor* paper, having a different color on each side, and fold it into the desired pattern of two colors. For example, John Montroll's book *Origami Inside-Out* [8] is entirely about such models. Taichiro Hasegawa [6] has designed an entire alphabet, including lower and upper case letters as well as punctuation. One origami designer, Toshikazu Kawasaki, has looked at the special case of *iso-area foldings*, that is, foldings that use equal amounts of both colors [7, p. 96]. See Figure 1.

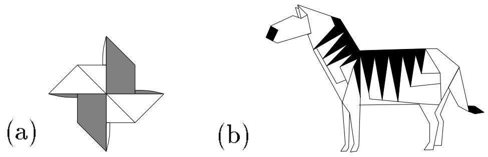

Figure 1: *(a) Iso-area pinwheel. (b) Zebra by John Montroll.*

Here we solve both of these problems with the following theorem; a more complete proof can be found in [4].

Theorem 1 *Given any polygonal region (polygon with holes) that is subdivided into polygonal regions each colored in one of two colors, there is a flat folding of a sufficiently large piece of bicolor paper that achieves this pattern.*

An important corollary of this result is that every polyhedron can be *wrapped* by a sufficiently large piece of paper. We can fill each face using Theorem 1, and connect between faces using the bridge gadget described below.

Our approach is to use *strips*, that is, arbitrarily long rectangles of paper. Note that a strip can be folded from any shape of paper; for example, we can accordion fold a square into a strip.

Our method is based on the idea of the *straight skeleton* [1] of a polygonal region. The straight skeleton is defined to be the trajectories of the boundary vertices during a *shrinking* process. This consists of insetting each boundary vertex so that every shrunken boundary edge is parallel to the original, and the distance between the shrunken and original boundary edges is the same over all boundary edges (at a particular time). Whenever the boundary becomes nonsimple, we recursively shrink the subregions.

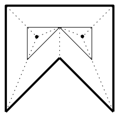

Figure 2: *Illustration of the straight skeleton and skeleton rings.*

During this shrinking process, the boundary changes in topology at finitely many times. At these points, the boundary forms *skeleton walls*, and the regions between them form *skeleton rings*; see Figure 2. More generally, a *ring* is a pair of walls such that shrinking one results in the other. Conceptually, the walls are parallel to each other except at turns, and have constant "width" all around. Note that the skeleton rings can be computed in $O(n^2)$ time from the straight skeleton, which can be computed in $O(n^{17/11+\epsilon})$ time [5].

Our basic algorithm is as follows. We compute the skeleton rings in each unicolor polygonal region. Define the *skeleton dual* of the two-color pattern to be the graph with vertices corresponding to skeleton rings, and edges between two vertices corresponding to rings that share a wall edge. Because the two-color pattern is a polygonal region, the skeleton dual is connected, and hence we can find a spanning tree.

We use this spanning tree as a road map for our construction. We perform a depth-first traversal of the tree, starting from an arbitrary root. At each new node we traverse, we construct the corresponding ring. Hence, we only need to show how to construct a skeleton ring, and how to connect between skeleton rings with an optional color change.

We choose to make the strip's width at most the width of the thinnest skeleton ring. This allows us to cover each skeleton ring by a collection of (potentially overlapping) *strip rings*, that is, rings with the width of the strip. We can construct such rings using *turn gadgets*, shown in Figure 3. A turn gadget consists of two main folds, followed by $\lceil \pi/2\theta \rceil - 1$ extra folds to "sharpen" the angle θ. Note that the turn gadgets cannot overlap each other, so the foldings cannot interfere with each other. Otherwise, the boundary would change topologically in the middle of the strip ring, which cannot occur because the strip

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ring is contained within a skeleton ring. Thus, we can combine turn gadgets to form a desired strip ring.

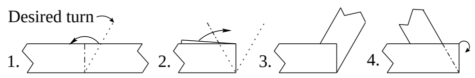

Figure 3: *(1–3) Folding a turn gadget for* $\theta \in [\pi/2, 3\pi/2]$. *(4) Extra folds for sharper angles.*

To complete the construction, we need to show how to combine strip rings in three different ways: between overlapping strip rings, between touching strip rings possibly in different skeleton rings, and between strip rings with different colors. In all cases, we take an arbitrary edge shared by the two strip rings; for overlapping rings, this "edge" has some thickness. We *bridge* at an appropriate joining place along this edge using two right-angle turns in the same direction; see Figure 4. This bridge forms a rectangle of extra paper that may fall outside the two strips. By accordion folding the rectangle sufficiently thin, and by choosing an appropriate joining place, this problem can be avoided.

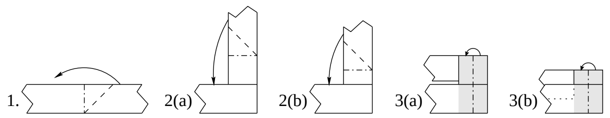

Figure 4: *Folding a bridge gadget (a) without overlap and (b) with overlap. Step 3 shows how to thin the rectangle of extra paper (light gray). Mountain [valley] folds are drawn in dot-dashed [dashed] lines.*

Finally, Figure 5 shows how to reverse the color of the strip in between the two right-angle turns.

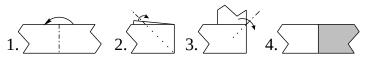

Figure 5: *Folding an angle-*π *color-reversal gadget.*

This completes our method for constructing an arbitrary two-color pattern formed by the union of polygonal regions. This preliminary result leaves open whether the number of folds can be bounded by a function of the input size. We conjecture that it is impossible to bound the number of folds by a function of the number of vertices, because sharp angles require a certain amount of "wrapping." However, to encode these sharp angles in the input requires many bits, so bounds in terms of the input size may be possible.

In the full version of this paper [4], we describe two other methods for solving this problem. The first method generalizes the notion of rings to make the minimum required strip width bounded by the minimum feature size (the minimum distance between two nonincident boundary edges) times the sine of half the sharpest angle.

The second method computes a Hamiltonian triangulation of the polygonal region using $O(n)$ Steiner points [2], and covers each triangle in turn along a Hamiltonian path in the dual. We cover each triangle by zig-zagging parallel to the edge e connecting this triangle to the next one along the path, starting the zig-zag at the vertex opposite to e. This forms some excess paper outside the triangle, which we can simply fold underneath the triangle. We choose the parity of the zig-zagging so that we end up at the appropriate vertex for the next triangle (opposite the edge connecting to the following triangle). By making the strip sufficiently thin, the amount of paper used by this method can approximate the area of the polygonal region.

Let us conclude with a gadget (Figure 6) that allows us to vary the width of the strip. Each application of the gadget can reduce the strip's width by between zero and fifty percent, by adjusting the position of the second fold. Using this tool, we can imagine using a wide strip to fill the area quickly, and thinning the strip for tight turns where needed, effectively localizing the requirements on the strip width.

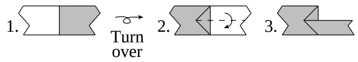

Figure 6: *Folding a strip-thinning gadget, starting from Step 4 of Figure 5.*

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