CSCI 0220 **Discrete Structures and Probability** B. Raphael

Homework 4

Solution Key

All homeworks are due at 1:00pm in the CS22 bin on the CIT second floor, opposite the elevators.

Write your full name *and the problem number on each piece of paper you hand in and then staple.*

Reading: S. Epp, sections 4.4; 5.1 – 5.4

Problem 4.1

Prove the following using induction:

$7 \mid 2^{3n} - 1, \forall n \geq 1$

Proof: By Induction on n. Let $P(k)$ be the predicate that $7 \mid 2^{3k} - 1$.

Base Case: When $k = 1$, $2^{3k} - 1 = 2^3 - 1 = 8 - 1 = 7$ and we know $7 \mid 7$. Therefore $P(1)$ is true.

Inductive Hypothesis: Assume $P(k)$ in order to prove $P(k+1)$. We must show that $7 \mid 2^{3(k+1)} - 1$, assuming that $7 \mid 2^{3k} - 1$. A little algebra helps:

$$\begin{aligned}
2^{3(k+1)} - 1 &= 2^{3k+3} - 1 \\
&= 2^{3k}2^3 - 1 \\
&= 2^{3k} * 8 - 1 \\
&= 8 * 2^{3k} - 1 \\
&= 8 * 2^{3k} - (8 - 7) \\
&= 8 * 2^{3k} - 8 + 7 \\
&= 8(2^{3k} - 1) + 7
\end{aligned}$$

We know by the induction hypothesis that $7 \mid 2^{3k} - 1$. We also know that 7 divides itself. Since 7 divides both terms of the expression, it must divide the sum of the terms, so we have

$$7 \mid 8(2^{3k} - 1) + 7$$

which, since $8(2^{3k} - 1) + 7 = 2^{3(k+1)} - 1$ by our algebra above, implies that

$$7 \mid 2^{3(k+1)} - 1$$

.

Therefore, we have proved, using mathematical induction, that $7 \mid 2^{3n} - 1$, $\forall n \geq 1$.

Problem 4.2

An n-player tournament *is a collection of two-player games such that each of the n players competes with each of the other players exactly once. (Thus there are $\frac{n(n-1)}{2}$ games). A player can either win or lose in a single game (there are no ties). At least two players are needed to start a tournmaent, otherwise there would be no point to having the tournament.*

Prove that in any n-player tournament, if there is no player that wins all his/her matches, then there is a non-empty collection of players $p_1, p_2, \ldots, p_k$ (for some $k \leq n$), such that player p_i beats player p_{i+1} for $1 \leq i < k$, and player p_k beats player p_1.

We will use induction. The inductive statement is

$P(n)$ = "In an n-player tournament where there is no player that beats everybody else, there exists a natural number $k \leq n$ and a collection of players $p_1, \ldots, p_k$ such that p_i beats p_{i+1} and p_k beats p_1".

Base case: The base case for $n = 2$ is true since there is always a player who beats everyone else.

Inductive step: Assume that $P(n)$ is true in order to show that $P(n+1)$ is true.

Consider a (n+1)-player tournament where there is no player that beats everybody else. Take any player out (call him/her p_1) and consider the remaining n-player tournament. There are two cases:

1) There is no player that beats everybody else.

2) There is some player that beats everybody else.

If the case is (1), then by the inductive hypothesis there is a cycle of players so we are done.

If the case is (2), then there is some player (say p_2) that beats everybody else in the n-player tournament. Then, because of the hypothesis that no one beats everybody in the $(n+1)$-player tournament, we have that p_1 beats p_2. Also, because of the same hypothesis, there is some player (call him/her

p_3) that beats p_1. Furthermore p_3 is beaten by p_2. So we have found a number $k = 3$ and a cycle $p_1 \rightarrow p_2 \rightarrow p_3 \rightarrow p_1$ of players that satisfy the inductive statement.

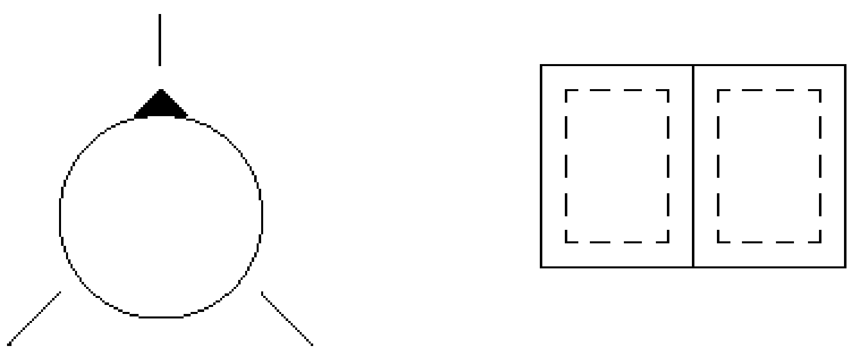
Note that what we have proved here is the somewhat stronger statement that in any tournament where there is no player that beats everybody, there exist a set of three players p_1, p_2, p_3, such that p_1 beats p_2, p_2 beats p_3, and p_3 beats p_1.

Problem 4.3

ACME agents have confiscated a safe that they think holds important V.I.L.E. documents. See figure below. On the front of the safe is a dial that can be turned to three valid positions. The dial is connected to a large digital display that has the amazing property of being able to display arbitrarily large numbers.

The dial starts in the upward position, and each time the dial is turned one "click" in either direction*, the display's inner count increases by one (it starts at "0"). However, the display only shows the count when the dial is in the upward position. This means, for example, that it is imposible to ever get the number "1" to display, since the dial must end pointing up.*

You can turn the dial in either direction as much as you want. Every time it goes back to its original position (the upward position), the display shows how many clicks happened.



It is easy to see that we can make the number "3" appear on the display: we simply turn the dial three clicks in one direction. As we've already said, though, it's impossible to make the number "1" appear.

The ACME Detective Agency wants to know whether all other positive integers can be made to appear on the display. Justify your answer.

Note: *The numbers only increase; while you can turn the dial in either direction, both directions just increase the number, and there is no way to make the numbers decrease. Remember that the display can hold an arbitrarily large number.*

Let n be an integer greater than 1.

The dial can be turned one click counterclockwise and then one click clockwise to produce the number 2. Simply turning the dial around one full turn will produce the number 3. Using combinations of these two methods, it seems reasonable that every number 2 and up could be reached.

Proof. By strong induction.

Let $P(n)$ be the statement "n can be made to show on the dial."

Basis Steps: $P(2)$ and $P(3)$.

As explained above, the dial can be turned one click counterclockwise and then one click clockwise to produce the number 2. Simply turning the dial around one full turn will produce the number 3.

Inductive Step: Let n be an integer greater than or equal to 3. Assume that $\forall k \in \{2, 3, \ldots, n\}, P(k)$. We need to show $P(n+1)$ (that $n+1$ can be made to show on the dial).

Since $n > 3$, $n - 1 > 1$. Applying the inductive hypothesis to $n - 1$, we conclude that $n - 1$ can be made to show on the dial.

By making $n - 1$ show on the dial, then turning the dial one click counterclockwise and one click clockwise, $n + 1$ will show on the dial.

Alternative proof. By strong induction.

Let $P(n)$ be the statement "n can be represented as $n = 2k + 3t$ where $k, t \in \mathbb{Z}^{\geq 0}$."

Basis Steps: $P(2)$ and $P(3)$.

For $k = 1$ and $t = 0$, $n = 2(1) + 3(0) = 2$.
For $k = 0$ and $t = 1$, $n = 2(0) + 3(1) = 3$.

Inductive Step:
Let n be an integer greater than or equal to 3.
Assume that $\forall k \in \{2, 3, \ldots, n\}, P(k)$.

We need to show $P(n+1)$ (that $n+1$ can be represented as $n+1 = 2k+3t$ where $k, t \in \mathbb{Z}^{\geq 0}$).

We know $P(n-1)$ by the inductive hypothesis. So $n-1 = 2a+3b$ where $a, b \in \mathbb{Z}^{\geq 0}$. Adding 2 to both sides, we get $n+1 = 2(a+1)+3b$. So if we let $k = a+1$ and $t = b$, we can represent $n+1$ in the form $2k+3t$.

We have completed the induction—all integers n greater than 1 can be represented as $2k+3t$, where k and t are nonnegative integers.

We can make n show on the dial by turning the dial one click counterclockwise and one click clockwise k times, then turning the dial all the way around t times.

Problem 4.4

Suppose that $s_0, s_1, s_2, \ldots$ is a sequence defined as follows

$$\begin{aligned} s_0 &= 12 \\ s_1 &= 29 \\ s_k &= 5(s_{k-1}) - 6(s_{k-2}) \end{aligned}$$

where $k \geq 2$.

Prove that, for $n \geq 0$, $s_n = 5(3^n) + 7(2^n)$.

Proof: The proof is by strong induction. Let $P(n)$ be the predicate that the formula $s_n = 5(3^n) + 7(2^n)$ is true for n.

Base Case: For $n = 2$, the explicit equation gives

$$s_2 = 5(3^2) + 7(2^2) = 5 * 9 + 7 * 4 = 45 + 28 = 73$$

which is the same result that we get using the recurrence relation

$$s_2 = 5 * s_1 - 6 * s_0 = 5 * 29 - 6 * 12 = 145 - 72 = 73$$

Induction Hypothesis: For $n \geq 2$ assume that the explicit equation gives the correct answer in order to prove that the correct answer is produced using the equation on $n+1$. The goal is to show that

$$s_{n+1} = 5(3^{n+1}) + 7(2^{n+1})$$

Starting with the recurrence relation and manipulating produces the desired result:

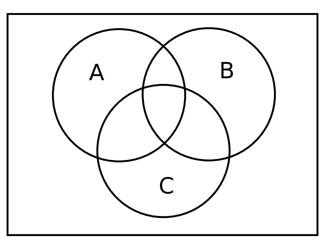
$$\begin{aligned}
s_{n+1} &= 5(s_n) - 6(s_{n-1}) \\
&= 5(5(3^n) + 7(2^n)) - 6(5(3^{n-1}) + 7(2^{n-1})) \\
&= 5 * 5(3^n) + 5 * 7(2^n) - 6 * 5(3^{n-1}) - 6 * 7(2^{n-1}) \\
&= 5 * 5(3^n) + 5 * 7(2^n) - 2 * 5(3^n) - 3 * 7(2^n) \\
&= (5-2) * 5(3^n) + (5-3) * 7(2^n) \\
&= 3 * 5(3^n) + 2 * 7(2^n) \\
&= 5(3^{n+1}) + 7(2^{n+1})
\end{aligned}$$

In the second step we recursively plug in the recurrence relation. In the third step we distribute terms. In the fourth we pull out a 3 and a 2 respectively out of the front of the last two terms and increment the exponent. In the fifth and sixth we combine terms, and in the last we push another 3 and 2 into the exponents.

This proves that for all $n \geq 2$, $P(2) \wedge P(3) \wedge \ldots P(n) \Rightarrow P(n+1)$. By strong induction $P(n)$ is true for all $n \geq 8$.

Problem 4.5

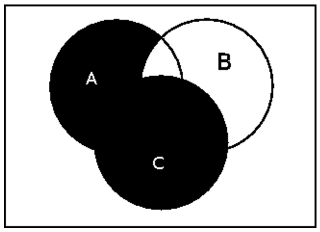
Shade the area of the following venn diagram which corresponds to each given set:

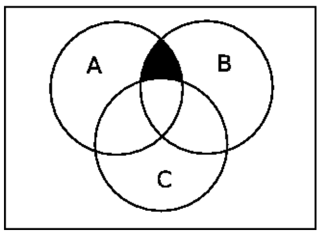


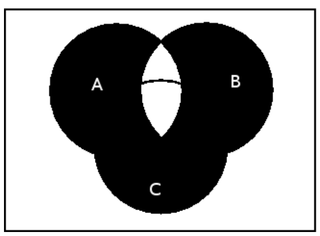
a. $(A - B) \cup C$

b. $(A \cap B) - C$

c. $((A \cup C) \cup (B \cup C)) \cap (A \cap B)^c$

a. 

b. 

c. 

Problem 4.6

Let A, B, C $\subseteq$ U. Prove or disprove that (A-B) $\subseteq$ C if and only if *(A-C) $\subseteq$ B.*

(A-B) $\subseteq$ C if and only if (A-C) $\subseteq$ B.

Suppose $(A - B) \subseteq C$. Then, $\forall a \in A \notin B, a \in C$. Then, $A - C - B = \emptyset$. Then, $\forall x$, if $x \in A$ and $x \notin C$, i.e. $x \in A - C$, then, since $A - C - B = \emptyset$, $x \in B$. Thus, $(A - C) \subseteq B$.

The converse simply results by swapping B and C in the above proof for essentially the same statement. This doesn't need to be shown explicitly, but it can't hurt.

The following problem is* non-collaborative*—discuss it with no one but the professor and the TAs.

Non-collaborative Problem 4.7

Given sets:

$A = \{1, 4, 6, 7, 12\}$
$B = \{2, 4, 6, 8, 10, 12\}$
$C = \{3, 4, 6, 8, 12\}$

Give:

a. The list of elements in $(A \cap B^c) \cap C^c$

b. The cardinality of the power set of $(B \cup C) \cap A^c$

c. The list of members of the power set of $(A \cap B) \cap C$

a. $\{1, 7\}$

b. $(B \cup C) \cap A^c = \{2, 3, 8, 10\}$
$2^4 = 16$

c. $\{\{\}, \{4\}, \{6\}, \{12\}, \{4, 6\}, \{4, 12\}, \{6, 12\}, \{4, 6, 12\}, \}$