CSCI 0220 **Discrete Structures and Probability** B. Raphael

Homework 4

Due: Friday, 29 Feb 2008

Reading: S. Epp, sections 4.4; 5.1 – 5.4

Problem 4.1

Prove the following using induction:

$7 \mid 2^{3n} - 1, \forall n \geq 1$

Problem 4.2

An *n-player tournament* is a collection of two-player games such that each of the n players competes with each of the other players exactly once. (Thus there are $\frac{n(n-1)}{2}$ games). A player can either win or lose in a single game (there are no ties). At least two players are needed to start a tournmaent, otherwise there would be no point to having the tournament.

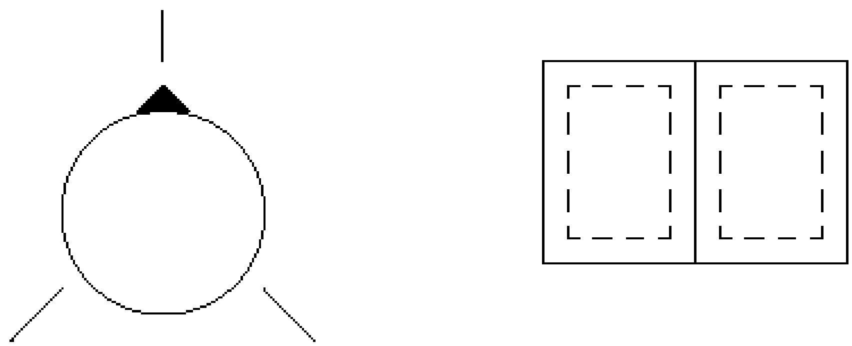
Prove that in any n-player tournament, if there is no player that wins all his/her matches, then there is a non-empty collection of players $p_1, p_2, \ldots, p_k$ (for some $k \leq n$), such that player p_i beats player p_{i+1} for $1 \leq i < k$, and player p_k beats player p_1.

Problem 4.3

ACME agents have confiscated a safe that they think holds important V.I.L.E. documents. See figure below. On the front of the safe is a dial that can be turned to three valid positions. The dial is connected to a large digital display that has the amazing property of being able to display arbitrarily large numbers.

The dial starts in the upward position, and each time the dial is turned one "click" *in either direction*, the display's inner count increases by one (it starts at "0"). However, the display only shows the count when the dial is in the upward position. This means, for example, that it is imposible to ever get the number "1" to display, since the dial must end pointing up.

You can turn the dial in either direction as much as you want. Every time it goes back to its original position (the upward position), the display shows how many clicks happened.



It is easy to see that we can make the number "3" appear on the display: we simply turn the dial three clicks in one direction. As we've already said, though, it's impossible to make the number "1" appear.

The ACME Detective Agency wants to know whether all other positive integers can be made to appear on the display. Justify your answer.

Note: The numbers only increase; while you can turn the dial in either direction, both directions just increase the number, and there is no way to make the numbers decrease. Remember that the display can hold an arbitrarily large number.

Problem 4.4

Suppose that $s_0, s_1, s_2, \ldots$ is a sequence defined as follows

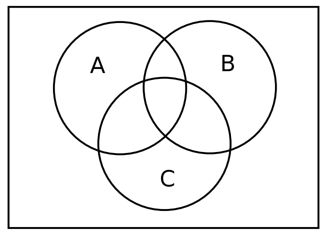
$$\begin{aligned} s_0 &= 12 \\ s_1 &= 29 \\ s_k &= 5(s_{k-1}) - 6(s_{k-2}) \end{aligned}$$

where $k \geq 2$.

Prove that, for $n \geq 0$, $s_n = 5(3^n) + 7(2^n)$.

Problem 4.5

Shade the area of the following venn diagram which corresponds to each given set:



a. $(A - B) \cup C$

b. $(A \cap B) - C$

c. $((A \cup C) \cup (B \cup C)) \cap (A \cap B)^c$

Problem 4.6

Let A, B, C $\subseteq$ U. Prove or disprove that (A-B) $\subseteq$ C *if and only if* (A-C) $\subseteq$ B.

The following problem is *non-collaborative*—discuss it with no one but the professor and the TAs.

Non-collaborative Problem 4.7

Given sets:

$A = \{1, 4, 6, 7, 12\}$
$B = \{2, 4, 6, 8, 10, 12\}$
$C = \{3, 4, 6, 8, 12\}$

Give:

a. The list of elements in $(A \cap B^c) \cap C^c$

b. The cardinality of the power set of $(B \cup C) \cap A^c$

c. The list of members of the power set of $(A \cap B) \cap C$