

# Homework 12

*Due: Friday, 2 May 2008*

NOTE: You must justify all answers with a proof or a convincing argument.  
Reading: Probability Notes 3,4,5

## Problem 12.1

You and your friend are in a heated debate about which of two restaurants to go to for dinner. To settle the argument, you propose to flip a coin. Your friend objects on the grounds that the Canadian quarter in your pocket is biased. To prove her point, she tells you to flip the coin until 50 heads appear. If the number of tails that has appeared at this point is not 50, then she claims that the coin must be biased.

- Let  $X$  be the number of flips until  $n$  heads appear. Find  $Pr[X = i]$ .
- What is the probability that your friend will agree that the coin is not biased?
- Compute  $E[X]$ . What does the answer tell you about your friend's proposed experiment?

Suppose that instead of having a Canadian quarter in your pocket, you had a box containing 50 red balls and 50 blue balls. You propose to draw 10 balls, without replacement, from the box and if there are more red than blue, then your restaurant is the winner. Your friend objects again, since the box is opaque and she can't verify that there are exactly 50 balls of each color. So she asks you to draw balls until the 10th red ball appears. If there aren't exactly 10 blue balls at this time, your box must be biased.

- In the general case, suppose that there are  $m$  red and  $n$  blue balls in the box. Let  $X$  be the number of balls that are drawn until the  $r$ -th red for  $1 \leq r \leq m$ . Find  $Pr[X = i]$ .
- What is the probability that your friend will agree that the box experiment is unbiased.
- Compute  $E[X]$ . What does the answer say about your friend's proposed experiment?

**Problem 12.2**

Suppose that a random simple graph  $G$  is constructed from a set of  $n$  vertices by joining each pair of nonidentical vertices by an edge with probability  $p$ . For each vertex  $v$  let  $X_v$  equal the degree of  $v$ , so  $X_v$  is a random variable.

- What is  $Pr[X_v = k]$  for  $k \geq 0$ ?
- What is  $E[X_v]$ ?
- Suppose  $G$  has one million vertices and  $p = \frac{1}{2}$ . Use Chebyshev's inequality to give an upper bound on the probability that a vertex  $v$  has degree  $\geq 501000$  or  $\leq 499000$ ?

**Problem 12.3**

Two players A and B play the following game. A coin with probability  $p$  of turning up heads is tossed repeatedly. Player A wins the game if heads appears at least  $m$  times before tails has appeared  $n$  times; otherwise, player B wins the game. Find the probability that player A wins the game.

**Problem 12.4**

Nick and Jon are having a dispute at the ACME Crime Lab. They disagree over the rate that cars pass through an intersection outside ACME HQ. Nick claims that 25 cars pass through the intersection every hour. Jon disagrees, arguing that only 10 cars pass through the intersection every hour. At least how many cars  $n$  must you observe passing through the intersection in one hour in order to claim that Nick is correct with 95% confidence? Prove your answer by defining random variable(s), calculating the probability that Nick is correct, and computing an answer for  $n$ . *Hint:* It may help in your calculation to first presume that Nick and Jon are equally likely to be correct.

**Problem 12.5**

Suppose the number of visitors to a museum each day has Poisson distribution with parameter  $\theta$ . Further assume that  $\theta$  varies from day to day according to a uniform distribution on  $[100, 300]$ . What is the

expected number of visitors on a randomly selected day? (Hint: Use  $E[E[X|Y]] = E[X]$ )

**Problem 12.6**

Suppose 1 out of 10,000 tickets is a winning ticket in a state lottery. How many tickets does one need to purchase so that he/she has at least 50% chance to have a winning ticket? (Use Poisson approximation)

**Problem 12.7**

Suppose John and Mary each tosses a fair coin twice. Let  $X$  be the number of heads John obtains and  $Y$  the number of heads Mary obtains.

- What is the distribution of  $X$ ? What is the distribution of  $Y$ ?
- What is the distribution of  $X + Y$ ?
- Given  $X + Y = 2$ , find the distribution and expected value of  $X$ .