

Homework 2

Solution Key

All homeworks are due at 1:00pm in the CS22 bin on the CIT second floor, opposite the elevators.

Write your full name and the problem number on each piece of paper you hand in and then staple.

Reading: Textbook sections, Chapter 2 (except material on Tarski's World and Prolog). Chapter 3, 3.1 – 3.3 (except unique factorization theorem).

Problem 2.1

Rewrite each of these statements so that negations are applied exclusively to predicates (that is, so that no negation precedes a quantifier or an expression involving logical connectives).

$$(a) \sim \forall x \forall y P(x, y)$$

$$= \exists x \exists y \sim P(x, y)$$

$$(b) \sim \forall y \exists x P(x, y)$$

$$= \exists y \forall x \sim P(x, y)$$

$$(c) \sim \forall y \forall x (P(x, y) \vee Q(x, y))$$

$$= \exists y \exists x \sim (P(x, y) \vee Q(x, y))$$

$$= \exists y \exists x (\sim P(x, y) \wedge \sim Q(x, y))$$

$$(d) \sim (\exists x \exists y \sim P(x, y) \wedge \forall x \forall y Q(x, y))$$

$$= (\sim \exists x \exists y \sim P(x, y)) \vee (\sim \forall x \forall y Q(x, y))$$

$$= (\forall x \forall y \sim \sim P(x, y)) \vee (\exists x \exists y \sim Q(x, y))$$

$$= (\forall x \forall y P(x, y)) \vee (\exists x \exists y \sim Q(x, y))$$

$$(e) \sim \forall x (\exists y \forall z P(x, y, z) \wedge \exists z \forall y P(x, y, z))$$

$$\begin{aligned} &= \exists x \sim (\exists y \forall z P(x, y, z) \wedge \exists z \forall y P(x, y, z)) \\ &= \exists x (\sim \exists y \forall z P(x, y, z) \vee \sim \exists z \forall y P(x, y, z)) \\ &= \exists x (\forall y \exists z \sim P(x, y, z) \vee \forall z \exists y \sim P(x, y, z)) \end{aligned}$$

Problem 2.2

Give the inverse, converse, and contrapositive of the following statements. Then simplify such that negations are applied exclusively to predicates (that is, so that no negation precedes a quantifier or an expression involving logical connectives).

a.

$$\forall x, \exists y, P(x) \wedge Q(x) \Rightarrow P(y)$$

b.

$$\forall z, (P(z) \wedge Q(z)) \vee (\exists x, P(x) \vee Q(x)), \Rightarrow S(z)$$

a.

$$\forall x, (P(x) \wedge Q(x)) \Longrightarrow \exists y, P(y)$$

Inverse:

$$\forall x, [\sim (P(x) \wedge Q(x)) \Longrightarrow \sim (\exists y, P(y))]$$

$$\forall x, [(\sim P(x) \vee \sim Q(x)) \Longrightarrow \forall y, \sim P(y)]$$

Converse:

$$\forall x, [\exists y, P(y) \Longrightarrow (P(x) \wedge Q(x))]$$

Contrapositive:

$$\forall x, [\sim (\exists y, P(y)) \Longrightarrow \sim (P(x) \wedge Q(x))]$$

$$\forall x, [\forall y, \sim P(y) \Longrightarrow \sim P(x) \vee \sim Q(x)]$$

b.

Inverse:

$$\forall z, \sim [(P(z) \wedge Q(z)) \vee (\exists x, P(x) \vee Q(x))] \implies \sim S(z)$$

$$\forall z, [\sim (P(z) \wedge Q(z)) \wedge \sim (\exists x, P(x) \vee Q(x))] \implies \sim S(z)$$

$$\forall z, [(\sim P(z) \vee \sim Q(z)) \wedge (\forall x, \sim P(x) \wedge \sim Q(x))] \implies \sim S(z)$$

Converse:

$$\forall z, S(z) \implies [(P(z) \wedge Q(z)) \vee (\exists x, P(x) \vee Q(x))]$$

Contrapositive:

$$\forall z, \sim S(z) \implies \sim [(P(z) \wedge Q(z)) \vee (\exists x, P(x) \vee Q(x))]$$

$$\forall z, \sim S(z) \implies [\sim (P(z) \wedge Q(z)) \wedge \sim (\exists x, P(x) \vee Q(x))]$$

$$\forall z, \sim S(z) \implies [(\sim P(z) \vee \sim Q(z)) \wedge (\forall x, \sim P(x) \wedge \sim Q(x))]$$

Problem 2.3

*Vampires and werewolves don't get along very well. This is partially because while the trustworthy werewolves always tell the truth, their sinister blood-sucking enemies **always** lie. Unfortunately, both vampires and werewolves look just like humans under ordinary conditions.*

Anyway, on the way to the CIT you run into three perfectly ordinary-looking humans who block your path. Let's just call them A, B, and C. You have the following conversation:

A: "Either B is a vampire, or C and I are the same."

B: "Oh yeah? If I'm a werewolf, then so is C."

C: "Well, I'm positive that A is a vampire."

Try as you might, they won't let you pass unless you correctly guess what each one of them is. You're sure that each one is either a vampire or a werewolf, and you also get the feeling that if you guess wrong, the result will be very messy.

What are the identities of the three beings blocking your path? Show your work.

We start with the following statements:

a : A is a werewolf.

b : B is a werewolf.

c : C is a werewolf.

Now, we have to model the truth conditions imposed by the conversation. They are as follows:

- $a \equiv (\tilde{b} \vee (a \wedge c) \vee (\tilde{a} \wedge \tilde{c}))$
- $b \equiv (b \rightarrow c)$
- $c \equiv \tilde{a}$

Let's start with the second equivalence. If b is true, then $b \rightarrow c$. If b is false, then $(b \rightarrow c)$, or $b \wedge \tilde{c}$. But if b is false, then $b \wedge \tilde{c}$ cannot be true, so b must be true.

Since b is true, we now know that $b \rightarrow c$ is true, so c is also true. Since c is true, that means a must be false. Just to make sure everything is right, we see that if a is false, then we know that $\tilde{b} \vee (a \wedge c) \vee (\tilde{a} \wedge \tilde{c})$ is false. That means that \tilde{b} is false (which it is), $a \wedge c$ is false (which it is), and $\tilde{a} \wedge \tilde{c}$ is false (which it is). Thus, we know that a is false, b is true, and c is true. So A is a vampire, and B and C are werewolves.

Problem 2.4

Let $S(x, y)$ be the statement “ x can bite y ,” where the domain is all people. Use predicates and quantifiers to express the following statements:

Define domains. D = all people. V = Vampires. W = Werewolves

- a. “All Vampires can be bitten by somebody.”

$$\forall x \in V, \exists y \in D, S(y, x)$$

- b. “There is exactly one person whom everybody can bite.”

$$\exists x \in D(\forall z \in D, S(z, x) \wedge \forall y [y \neq x \implies \exists z \in V, \neg S(z, y)])$$

- c. Express the following statement by two different but equivalent logical expressions using what you know about predicate logic:
 “No one can bite both a Werewolf and a Vampire.”

$$\neg [\exists x \in D, \exists y \in V S(x, y) \wedge \exists z \in W S(x, z)]$$

$$\forall x \in D, [\forall y \in V, \neg S(x, y) \vee \forall z \in W, \neg S(x, z)]$$

- d. Now demonstrate that your two statements in part c are logically equivalent.

We begin with the first statement.

$$\neg [\exists x \in D, \exists y \in V S(x, y) \wedge \exists z \in W S(x, z)]$$

Negating the first existential quantifier, we get

$$\forall x \in D, \neg [\exists y \in V S(x, y) \wedge \exists z \in W S(x, z)]$$

Using De Morgan’s law, we can rewrite this as

$$\forall x \in D, [\neg(\exists y \in V S(x, y)) \vee \neg(\exists z \in W S(x, z))]$$

Negating the remaining quantifiers, we get

$$\forall x \in D, [\forall y \in V \neg S(x, y) \wedge \forall z \in W \neg S(x, z)]$$

which is the second statement. We have thus proved that the two statements are logically equivalent.

Problem 2.5

Theorem $\forall a, b, c \in \mathbb{Z}, (a \mid b \wedge a \nmid c) \Rightarrow a \nmid (b + c)$ (\nmid means “not divisible”)

- Prove the above by contradiction.
- Prove it by contraposition.

Proof by contradiction: Suppose $a \nmid (b + c)$ is not true, that is, $a \mid (b + c)$. Then $\exists k \in \mathbb{Z}$ such that $ak = (b + c)$. We also know that $a \mid b$, which means $\exists i \in \mathbb{Z}$ such that $ai = b$. Thus we have $ak = (ai + c)$. Dividing both sides by a , we have $k = i + \frac{c}{a}$, which leads to $(k - i) = \frac{c}{a}$, and thus $c = a(k - i)$. However, we know that $a \nmid c$, which is a contradiction. Thus $a \nmid (b + c)$.

Proof by contraposition: The contrapositive is

$$\forall a, b, c \in \mathbb{Z}, \quad a \mid (b + c) \rightarrow \neg(a \mid b \wedge a \nmid c)$$

or equivalently

$$\forall a, b, c \in \mathbb{Z}, \quad a \mid (b + c) \rightarrow (a \nmid b \vee a \nmid c)$$

Consider two cases:

- a. $a \nmid b$: then the implication is trivially true.
- b. $a \mid b$: then $\exists m \in \mathbb{Z}$ such that $am = b$. Also, by assumption $\exists n \in \mathbb{Z}$ such that $an = b + c$. By substitution,

$$\begin{aligned} an &= am + c \\ an - am &= c \\ a(n - m) &= c \end{aligned}$$

$n - m$ is an integer, so $a \mid c$ and the implication is true.