CSCI 0220 **Discrete Structures and Probability** F. Preparata

Homework 4

Solution Key

All homeworks are due at 1:00pm in the CS22 bin on the CIT second floor, opposite the elevators.

Write your full name *and the problem number on each piece of paper you hand in and then staple.*

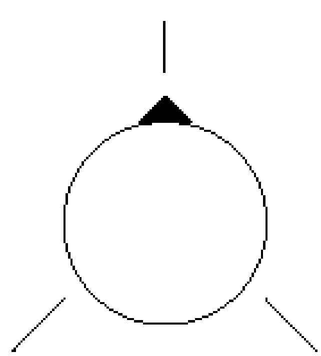
Reading: *Chapter 3: Section 3.8. Chapter 4: Section 4.5 pp. 619-620.*

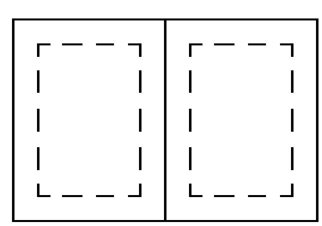
Problem 4.1

Your werewolf friend Horowitz recently acquired a strange artifact from a vampire's clutches. It looks a little bit like a safe. On the front of the safe is a dial that can be turned to three valid positions. The dial is connected to a large digital display that has the amazing property of being able to display arbitrarily large non-negative integers.

The dial starts in the upward position, and each time the dial is turned one "click" in either direction*, the display's inner count increases by one (it starts at "0"). However, the display only shows the count when the dial is in the upward position. This means, for example, that it is imposible to ever get the number "1" to display, since the dial must end pointing up.*

You can turn the dial in either direction as much as you want. Every time it goes back to its original position (the upward position), the display shows how many clicks happened.





It is easy to see that we can make the number "3" appear on the display:

we simply turn the dial three clicks in one direction. As we've already said, though, it's impossible to make the number "1" appear.

Horowitz wants to know whether all other positive integers can be made to appear on the display. He also wants you to justify your answer.

Note: *The numbers only increase; while you can turn the dial in either direction, both directions just increase the number, and there is no way to make the numbers decrease. Remember that the display can hold an arbitrarily large integer.*

Let n be an integer greater than 1.

The dial can be turned one click counterclockwise and then one click clockwise to produce the number 2. Simply turning the dial around one full turn will produce the number 3. Using combinations of these two methods, it seems reasonable that every number 2 and up could be reached.

Proof. By strong induction.

Let $P(n)$ be the statement "n can be made to show on the dial."

Basis Steps: $P(2)$ and $P(3)$.

As explained above, the dial can be turned one click counterclockwise and then one click clockwise to produce the number 2. Simply turning the dial around one full turn will produce the number 3.

Inductive Step: Let k be an integer greater than or equal to 3. Assume that for all n, $2 \leq n \leq k, P(n)$. We need to show $P(k+1)$ (that $k+1$ can be made to show on the dial).

Since $k > 3$, $k - 1 > 1$. Applying the inductive hypothesis to $k - 1$, we conclude that $k - 1$ can be made to show on the dial.

By making $k - 1$ show on the dial, then turning the dial one click counterclockwise and one click clockwise, $k + 1$ will show on the dial.

We've therefore shown that the statement is true for $n = 2$ and $n = 3$, and that it is true for $n = k + 1$, assuming it is true for all $2 \leq n \leq k$. Therefore the statement is true for all $n \geq 2$.

Problem 4.2

If you have \$3-*bills and a* \$5-*bills you can express any amount over* \$7. *Using induction, prove this theorem.*

Proof by strong induction:

Claim: $\forall n > 7$, n can be express as $3x + 5y$, where x and y are positive integers.

Base Case: $8 = 3(1) + 5(1)$, $9 = 3(3) + 5(0)$, $10 = 3(0) + 5(2)$, therefore the claim is true for $n = 8, 9, 10$.

Inductive Hypothesis: We assume that the claim is true for $n \leq k$:

$$\forall \ l \leq k \ \ (\exists x \exists y (l = 3x + 5y))$$

Inductive Step: Using our assumption, we prove that the claim is true for $n = k + 1$:

$$k + 1 = (k - 2) + 3 = (3x + 5y) + 3 = 3(x + 1) + 5y = 3x' + 5y$$

We've shown that the claim is true for $n = 8, 9, 10$, and that it is true for $n = k + 1$ given that it is true for all $n \leq k$, therefore the claim is true for all $n > 7$ by strong induction.

Problem 4.3

Prove by induction that

$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Basis: $n = 1$

$$\sum_{i=1}^{1} i^3 = 1$$

$$\left(\frac{1(1+1)}{2}\right)^2 = \left(\frac{2}{2}\right)^2 = 1^2 = 1$$

Therefore the statement is true for $n = 1$.

Inductive Step: Assume true for $n = k$; that is, assume that

$$\sum_{i=1}^{k} i^3 = \left(\frac{k(k+1)}{2}\right)^2$$

Show true for $n = k + 1$, thus showing that:

$$\begin{aligned}
\sum_{i=1}^{k+1} i^3 &= \left(\frac{(k+1)(k+2)}{2}\right)^2 \\
\sum_{i=1}^{k+1} i^3 &= \sum_{i=1}^{k} i^3 + (k+1)^3 \\
&= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 \\
&= \frac{k^2(k+1)^2 + 4(k+1)^3}{4} \\
&= \frac{(k+1)^2(k^2 + 4(k+1))}{4} = \frac{(k+1)^2(k^2 + 4k + 4)}{4} \\
&= \frac{(k+1)^2(k+2)^2}{4} = \left(\frac{(k+1)(k+2)}{2}\right)^2
\end{aligned}$$

We've proven that the statement is true for $n = 1$ and $n = k + 1$ assuming it's true for $n = k$. By induction, it is therefore true for all $n >= 1$.

Problem 4.4

Suppose that $h_0, h_1, h_2, h_3, \ldots$ is a sequence defined as follows: $h_0 = 1, h_1, = 2, h_2 = 3, h_k = h_{k-1} + h_{k-2} + h_{k-3}$ for all integers $k \geq 3$. Prove that $h_n \leq 3^n$ for all integers $n \geq 0$.

Proof: By the second form of induction
First I will prove the claim true for the following bases cases:

$$n = 0,\ h_0 = 1 = 1 \leq 3^0 = 1.$$
$$n = 1,\ h_1 = 1 + 2 = 3 \leq 3^1 = 3.$$
$$n = 2,\ h_2 = 1 + 2 + 3 = 6 \leq 3^2 = 9.$$

Now having show my bases cases to be true, now by induction assume that claim is true for $n = k$, namely that:

$$\forall k, 0 < k \leq n \text{ that } h_n \leq 3^n$$

Now to prove claim true using induction, we must show that $h_n \leq 3^n$ is true for $n = k + 1$. However by the premise we can write:

$$h_{k+1} = h_k + h_{k-1} + h_{k-2}$$

But by the inductive hypothesis we know that:

$$h_{k+1} \leq 3^k + 3^{k-1} + 3^{k-2}$$

Then it must also follow that:

$$h_{k+1} \leq 3^k + 3^k + 3^k \text{ because } 3^k \geq 3^{k-1} \forall k > 0.$$
$$h_{k+1} \leq 3(3^k)$$
$$h_{k+1} \leq 3^{k+1}.$$

Thus having shown the claim is true for $n = k + 1$, using the second form of induction, I have proven the claim is true.

Problem 4.5

The game Mini-nim is defined as follows: Some positive number of sticks are placed on the ground. Two players take turns removing one, two, or three sticks. The player to remove the last one loses. Use the second form of induction to show that the second player has a winning strategy if and only if the number of remaining sticks n equals $4k + 1$ for some $k \in \mathbb{Z}$.

Claim: The second player has a winning strategy if and only if the number of remaining sticks n equals $4k + 1$ for some $k \in \mathbb{Z}$.

Proof: By the second form of induction I will prove that the claim is true. First let us prove the statement that if player 2 has a winning strategy that this implies there are $n = 4k + 1$ for some $k \in \mathbb{Z}$ sticks remaining. Using the second form of induction I will prove the claim true. First consider the basis that there is $n = 1$ stick remaining. If this is the case it follows that:

$$1 = 4k + 1$$
$$0 = 4k$$
$$k = 0$$

Since $k = 0$ and $k \in \mathbb{Z}$ the claim is thus true for the basis. Now by the second form of induction assume that the claim is true, namely that player 2 having a winning strategy implies there are $n = 4k+1$ sticks remaining for $1 \leq z < n$. Now to complete this prove we must show that that the claim is true for $n = z + 1$. If there are $n = z + 1$ sticks remaining then player 1 is left with three options, he can either take 1, 2, or 3 sticks, which would then mean there can either be z, $z - 1$, or $z - 2$ stick remaining. However if there are z, $z - 1$, or $z - 2$ sticks remaining by the inductive hypothesis it must follow that there are some integers k_1, k_2, k_3 such that:

$$z = 4 * k_1 + 1$$
$$z - 1 = 4 * k_2 + 1$$
$$z - 2 = 4 * k_3 + 1$$

Thus having shown the claim true for the case where there are $n = z + 1$ sticks remaining the claim that player 2 having a winning strategy implies that there are $n = 4k + 1$ sticks remaining is thus proven true.

Now to complete my proof, I must show that if there are $n = 4k + 1$ sticks remaining this implies that player 2 has a winning strategy. To prove this true, I will invoke the second form of induction on the number of remaining sticks. First consider the following bases cases:

$n = 1$. The first player must remove the last stick, and therefore loses.
$n = 2$. The first player removes 1 stick, leaving one. The second player loses.
$n = 3$. The first player removes 2 sticks, leaving one. The second player loses.
$n = 4$. The first player removes 3 sticks, leaving one. The second player loses.

Thus having shown the claim true for the following base cases, now assume by the second form of induction for $1, ..., n$; that is, assume that for all numbers i of sticks $1 \leq i \leq n$ left at the beginning of the first player's turn, the second player has a winning strategy if and only if i can be expressed in the form $4k + 1$ for some $k \in \mathbb{Z}$. We must show that it is true for $n + 1$. We consider four cases.

- $n+1 = 4k+1$: Because the base case considered $k = 1$, we may assume that $k > 1$ and therefore $n+1 \geq 5$. The first player can remove 1, 2, or 3 sticks. Assume the player removes 1 stick. Then the second player sees $n = 4k$ sticks and wins by the assumption that the claim is true for n. Similarly, the second player sees $n - 1 = 4k - 1 = 4(k - 1) + 3$ sticks if the first player removes 2 sticks and $n - 2 = 4k - 2 = 4(k - 1) + 2$

sticks if the first player removes 3 sticks. By the inductive hypothesis, since the second player sees a number of sticks not equal to $4k + 1$ for any $k \in \mathbb{Z}$, and therefore wins by the assumptions that the claim is true for $n - 1$, $n - 2$. Therefore, the second player has a winning strategy.

- $n + 1 = 4k$: If first player removes 3 sticks, the second player sees $n - 2 = 4k - 3 = 4(k - 1) + 1$ sticks and loses by the assumption that the claim is true for $n - 2$.
- $n + 1 = 4k + 2$: If the first player removes 1 stick, the second player sees $n = 4k + 1$ sticks and loses by the assumption that the claim is true for n.
- $n + 1 = 4k + 3$: If the first player removes 2 sticks the second player sees $n - 1 = 4k + 1$ sticks and loses by the assumption that the claim is true for $n - 1$.

We have therefore proved that the inductive hypothesis is true, and the claim is true for all $n \geq 1$.