CSCI 0220 **Discrete Structures and Probability** F. Preparata

Homework 5

Solution Key

All homeworks are due at 1:00pm in the CS22 bin on the CIT second floor, opposite the elevators.

Write your full name *and the problem number on each piece of paper you hand in and then staple.*

Reading*: Chapter 5 (up to pg. 294, excluding "Halting Problem").*

Problem 5.1

Let A, B, C be sets, and assume they are all subsets of a universal set U. Using the **element method***, show that the following equivalences hold:*

a. $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$

b. $A \cup (B \cap (C - A)) = A \cup (B \cap C)$

c. $(A - B) \cap (B - C) \cap (A - C) = \emptyset$

a. Let $x \in (A - B) \cup (B - A)$. Then by definition, either $x \in A \wedge x \notin B$ or $x \in B \wedge x \notin A$. Thus, x is in A or B, but not both, thus $x \in (A \cup B) - (A \cap B)$.

   Now let $x \in (A \cup B) - (A \cap B)$. That means $x \in A$ or $x \in B$, but not both. So either $x \in (A - B)$ or $x \in (B - A)$, so $x \in (A - B) \cup (B - A)$.

b. Let $x \in A \cup (B \cap (C - A))$. Then either $x \in A$, or $x \in B \cap (C - A)$. If $x \in A$, then $x \in A \cup (B \cap C)$. Otherwise, $x \in B$ and $x \in (C - A)$, so $x \in C$, so $x \in B \cap C$, so $x \in A \cup (B \cap C)$.

   Now, let $x \in A \cup (B \cap C)$. If $x \in A$, then $x \in A \cup (B \cap (C - A))$. Thus, assume $x \notin A$. Then $x \in B \cap C$, so $x \in B$ and $x \in C$. Since $x \notin A$, $x \in (C - A)$, so $x \in B \cap (C - A)$. Thus, $x \in A \cup (B \cap (C - A))$.

c. Assume for the sake of contradiction that there is some $x \in (A - B) \cap (B - C) \cap (A - C)$. Then $x \in A$, $x \notin B$, $x \in B$, $x \notin C$, $x \in A$, and $x \notin C$. But x can't be both in and not in B! Contradiction, so there is no such x, so that set must be empty.

Problem 5.2

Let A, B, C be sets, and assume they are all subsets of a universal set U. Using **set algebra**, *show that the following equivalences hold:*

a. $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$

b. $A \cup (B \cap (C - A)) = A \cup (B \cap C)$

c. $(A - B) \cap (B - C) \cap (A - C) = \emptyset$

a.

$$\begin{aligned}(A - B) \cup (B - A) &= (A \cap B^c) \cup (B \cap A^c) \\ &= (A \cup B) \cap (A \cup A^c) \cap (B^c \cup B) \cap (B^c \cup A^c) \\ &= (A \cup B) \cap (A \cap B)^c \\ &= (A \cup B) - (A \cap B)\end{aligned}$$

b.

$$A \cup (B \cap (C - A)) = (A \cup B) \cap (A \cup (C - A)) = (A \cup B) \cap (A \cup C) = A \cup (B \cap C)$$

c.

$$(A - B) \cap (B - C) \cap (A - C) = (A \cap B^c) \cap (B \cap C^c) \cap (A \cap C^c) = B^c \cap B \cap \cdots = \emptyset \cap \cdots = \emptyset$$

Problem 5.3

Prove or disprove that $X \cap Z \subseteq Y \Leftrightarrow (X - Y) \cup (Y - Z) \subseteq Z^c$

We begin by proving two lemmas.
Lemma 1: $A \cup B \Leftrightarrow A \cap C^c \subseteq \emptyset$

$$\begin{aligned}A \cup B &\Leftrightarrow [x \in A \Rightarrow x \in C] \\ &\Leftrightarrow [x \in A \Rightarrow \sim (x \notin C)] \\ &\Leftrightarrow [x \in A \Rightarrow \sim (x \notin C^c)] \\ &\Leftrightarrow A \cap C^c = \emptyset\end{aligned}$$

Lemma 2: $A \cup B \subseteq C \Leftrightarrow A \subseteq C \wedge B \subseteq C$
$A \cup B \subseteq C \Rightarrow A \subseteq C \wedge B \subseteq C$
We know that, by the definitions of union and subset,

$$\forall x \in U,\ (x \in A \vee x \in B) \Rightarrow x \in C$$

This means that if x is an element of A, then X is an element of C. The same reasoning applies when x is an element of B. By definition, this means that $A \subseteq C \wedge B \subseteq C$.

$A \subseteq C \wedge B \subseteq C \Rightarrow A \cup B \subseteq C$
By definition,

$$A \subseteq C \wedge B \subseteq C \Rightarrow [(x \in A \Rightarrow x \in C) \wedge (x \in B \Rightarrow x \in C)]$$

Then if x is an element of A or x is an element of B, then x is an element of C; that is,

$$x \in A \cup B \Rightarrow x \in C$$

$$A \cup B \subseteq C$$

We are now ready to prove our original statement.
Claim: $X \cap Z \subseteq Y \Leftrightarrow (X - Y) \cup (Y - Z) \subseteq Z^c$
Proof:
First we must prove that

$$(X - Y) \cup (Y - Z) \subseteq Z^c \Rightarrow X \cap Z \subseteq Y$$

We can rewrite the first statement as $(X \cap Y^c) \cup (Y \cap Z^c) \subseteq Z^c$. By Lemma 2, we know that $X \cap Y^c \subseteq Z^c$ and $Y \cap Z^c \subseteq Z^c$. The second term gives us no new information; we know that the intersection of two sets is a subset of each. However, from the first, by Lemma 1 we know that $(X \cap Y^c) \cap Z = \emptyset$. By commutativity and associativity, we can rewrite this as $(X \cap Y) \cap Y^c = \emptyset$. Again by Lemma 1, we can then say that $X \cap Z \subseteq Y$. We have therfore proved that $(X - Y) \cup (Y - Z) \subseteq Z^c \Rightarrow X \cap Z \subseteq Y$.

Now we must prove that

$$X \cap Z \subseteq Y \Rightarrow (X - Y) \cup (Y - Z) \subseteq Z^c$$

By Lemma 1, our assumption can be written as $(X \cap Z) \cap Y^c = \emptyset$. Then $(X \cap Y^c) \cap Z = \emptyset$ and $X \cap Y^c \subseteq Z^c$. We know that $Y \cap Z^c \subseteq Z^c$, so by Lemma 2 we then have that $(X \cap Y^c) \cup (Y \cap Z^c) \subseteq Z^c$. Again, this can be rewritten as $(X - Y) \cup (Y - Z) \subseteq Z^c$, and we have proved that $(X - Y) \cup (Y - Z) \subseteq Z^c \Rightarrow X \cap Z \subseteq Y$.

We have demonstrated that both directions are true, and our proof is complete.

Problem 5.4

Let A, B, C $\subseteq$ U. Prove or disprove that (A-B) $\subseteq$ C if and only if *(A-C) $\subseteq$ B.*

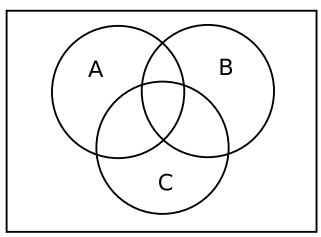
(A-B) $\subseteq$ C if and only if (A-C) $\subseteq$ B.

Suppose $(A-B) \subseteq C$. Then, $\forall a \in A \notin B, a \in C$. Then, $A - C - B = \emptyset$. Then, $\forall x$, if $x \in A$ and $x \notin C$, i.e. $x \in A - C$, then, since $A - C - B = \emptyset$, $x \in B$. Thus, $(A - C) \subseteq B$.

The converse simply results by swapping B and C in the above proof for essentially the same statement. This doesn't need to be shown explicitly, but it can't hurt.

Problem 5.5

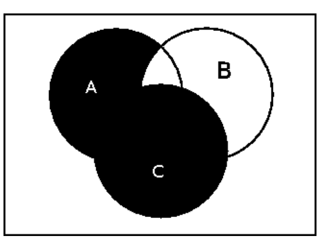
Shade the area of the following Venn diagram which corresponds to each given set (for each case, produce the corresponding instance of the Venn diagram):

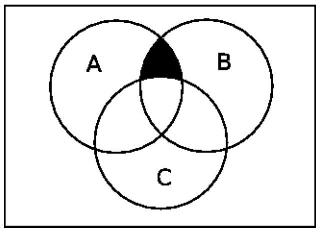


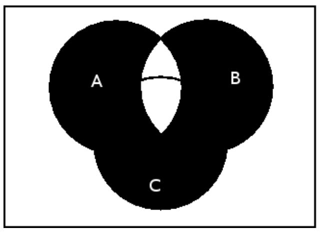
a. $(A - B) \cup C$

b. $(A \cap B) - C$

c. $((A \cup C) \cup (B \cup C)) \cap (A \cap B)^c$

a. 

b. 

c. 

Problem 5.6

Given sets:

$A = \{1, 4, 6, 7, 12\}$
$B = \{2, 4, 6, 8, 10, 12\}$
$C = \{3, 4, 6, 8, 12\}$

Give:

a. *The list of elements in* $(A \cap B^c) \cap C^c$

b. *The cardinality of the power set of* $(B \cup C) \cap A^c$

c. *The list of members of the power set of* $(A \cap B) \cap C$

a. $\{1, 7\}$

b. $(B \cup C) \cap A^c = \{2, 3, 8, 10\}$
$2^4 = 16$

c. $\{\{\}, \{4\}, \{6\}, \{12\}, \{4, 6\}, \{4, 12\}, \{6, 12\}, \{4, 6, 12\}, \}$