Homework 6

Due: 13 Mar 2009

All homeworks are due at 1:00pm in the CS22 bin on the CIT second floor, opposite the elevators.

Write your *full name* and the problem number on each piece of paper you hand in and then staple.

Reading: Chapter 10: 10.1, 10.2, 10.3, 10.4 (up to pp. 620), 10.5 (up to pp. 644).

Problem 6.1

For each of the following relations, R_1, R_2, and R_3 write whether they satisfy reflexivity, symmetry, and transitivity. If the relation satisfies the property, prove that it does. If it doesn't, give a counterexample.

For example, for the first relation, if you think it's transitive and reflexive but not symmetric, write that, show that the relation is transitive and reflexive, and give a counterexample to show that it is not symmetric.

- A is the set of all lines in the plane. R_1 is the relation of perpendicularity. For $L, L' \in A$, $(L, L') \in R_1 \Leftrightarrow L$ is perpendicular to L'.
- $\forall a, b \in \mathbb{N}, (a, b) \in R_2 \Leftrightarrow a \neq b$.
- $\forall a, b \in \mathbb{N}, (a, b) \in R_3 \Leftrightarrow \frac{a}{b} = 2^i$ for some integer $i \geq 0$.

Problem 6.2

For each of the following posets, draw a Hasse diagram showing the partial order relation.

a) The Power Set of a, b, c, d, where for any subsets A, B of a, b, c, d, $A \leq B \Leftrightarrow A \subseteq B$. (This is also called the **inclusion order**.)

b) Given the totally ordered sets $0, \ldots, n$ and $0, \ldots, m$ with the ordinary $\leq$ orders, consider the Cartesian product of these sets, with $(a, b) \leq (c, d) \Leftrightarrow (a \leq c) \wedge (b \leq d)$.

c) The set of finite sequences (strings) of the letters a, b. For example, *abbab* is such a sequence, as is ϵ, the empty sequence. $S_1 \preceq S_2$ iff S_1 is an *initial subsequence* of S_2 (also called the *prefix* of S_2); in other words, if n is the number of letters in S_1, then the first n letters of S_2 are exactly the sequence S_1.

Problem 6.3

For each of the following relations, prove whether it is an equivalence relation. If so, list the equivalence classes. Otherwise, display the relation graphically.

a. $R = \{(a, b) \mid a = b^2 \; mod \; 5\}$ on set $\{1, 2, 3...16\}$

b. $R = \{(a, b) \mid gcd(a, b) = 1\}$ (i.e., a and b are prime, having no common divisors except for 1.) for $\{1, 2, ...10\}$

c. $R = \{(a, b) \mid a = (2b)^2\}$ for $\{-5, -4, -3...5\}$

Problem 6.4

Let C be the set of the 16 2x2 chessboards whose squares are all colored either red or blue. Let R be a relation on C such that for two such chessboards, c_1 and c_2, $(c_1, c_2) \in R$ if and only if c_2 can be obtained from c_1 by a (finite) sequence of 90 degree rotations. (Imagine a nail in the center of the board, where the four squares meet. "One rotation," then, is accomplished by rotating the board $\pi/2$ or $-\pi/2$ about the nail.)

a. Show that R is an equivalence relation.

b. What are the equivalence classes of R?

Problem 6.5

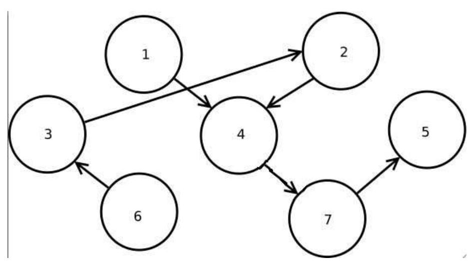
a. Let (S, R) be a poset. Show that (S, R^{-1}) is also a poset, where R^{-1} is the inverse relation of R. The poset (S, R^{-1}) is called the *dual* of (S, R).

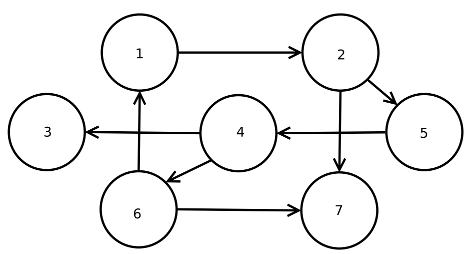
b. Suppose that $(S, \preceq_1)$ and $(T, \preceq_2)$ are posets. If $\preceq$ is defined by $(s,t) \preceq (u,v)$ if and only if $s \preceq_1 u$ and $t \preceq_2 v$ for $s, u \in S$, and $t, v \in T$, show that $(S \times T, \preceq)$ is a poset

Problem 6.6

Solve for each of the following:

- Can the graph be sorted topologically?
- If so, does there exist a unique sort?
- If there is more than one sort, enumerate all possible sorts.

a. 

b. 

c.