Homework 6

Solution Key

All homeworks are due at 1:00pm in the CS22 bin on the CIT second floor, opposite the elevators.

Write your full name *and the problem number on each piece of paper you hand in and then staple.*

Reading: *Chapter 10: 10.1, 10.2, 10.3, 10.4 (up to pp. 620), 10.5 (up to pp. 644).*

Problem 6.1

For each of the following relations, R_1, R_2, and R_3 write whether they satisfy reflexivity, symmetry, and transitivity. If the relation satisfies the property, prove that it does. If it doesn't, give a counterexample.

For example, for the first relation, if you think it's transitive and reflexive but not symmetric, write that, show that the relation is transitive and reflexive, and give a counterexample to show that it is not symmetric.

- *A is the set of all lines in the plane. R_1 is the relation of perpendicularity. For $L, L' \in A$, $(L, L') \in R_1 \Leftrightarrow L$ is perpendicular to L'.*
- $\forall a, b \in \mathbb{N}, (a, b) \in R_2 \Leftrightarrow a \neq b$.
- $\forall a, b \in \mathbb{N}, (a, b) \in R_3 \Leftrightarrow \frac{a}{b} = 2^i$ *for some integer* $i \geq 0$.

For counterexamples, any similar argument is fine.

- A is the set of all lines in the plane. R_1 is the relation of perpendicularity. For $L, L' \in A(L, L') \in R_1, \leftrightarrow L$ is perpendicular to L'.
  - **Not Reflexive.** Any line is a counterexample. A line cannot be perpendicular to itself because it makes a 180 degree angle with itself.

  - **Symmetric.** If one line makes a 90 degree angle with a second line, then the second line makes a 90 degree angle with the first as well.
  - **Not Transitive.** Two lines perpendicular to a third line are either parallel or the same line, so they cannot be perpendicular to each other.

- $\forall a, b \in \mathbb{N}, (a, b) \in R_2 \Leftrightarrow a \neq b$.

  - **Not Reflexive.** If R_2 was reflexive, for example, (4,4) would be in the relation. And $4 = 4$, which is in violation of $a \neq b$. Thus R_1 is not reflexive.
  - **Symmetric.** By commutativity, if $a \neq b$, then $b \neq a$.
  - **Not Transitive.** Consider (6,4) and (4,6) as pairs in the relation. If R_2 was transitive, then (6,6) would be in the relation, and since the relation is not reflexive as shown above, that is a contradiction. Thus, R_2 is not transitive.

- $\forall a, b \in \mathbb{N}, (a, b) \in R_3 \Leftrightarrow \frac{a}{b} = 2^i$ for some integer $i \geq 0$.

  - **Reflexive.** $\frac{x}{x} = 1, and 2^0 = 1$, a valid value of i.
  - **Not Symmetric.** Consider (6,3) as a pair in the relation. $\frac{6}{3} = 2^1$. If R_3 were symmetric, (3,6) would also be in the relation. $\frac{3}{6} = \frac{1}{2} = 2^{-1}$. Therefore i would equal -1, but i must be ≥ 0. Thus R_3 is not symmetric.
  - **Transitive.** Consider pairs (a,b) and (b,c) in the relation. $\frac{a}{b} = 2^i$ and $\frac{b}{c} = 2^j$. $\frac{a}{b} \times \frac{b}{c} = \frac{a}{c} = 2^{i+j}$, and since $i \geq 0$ and $j \geq 0$ since (a,b) and (b,c) are in the relation, $i + j \geq 0$. Thus, (a,c) is in the relation, and R_3 is transitive.

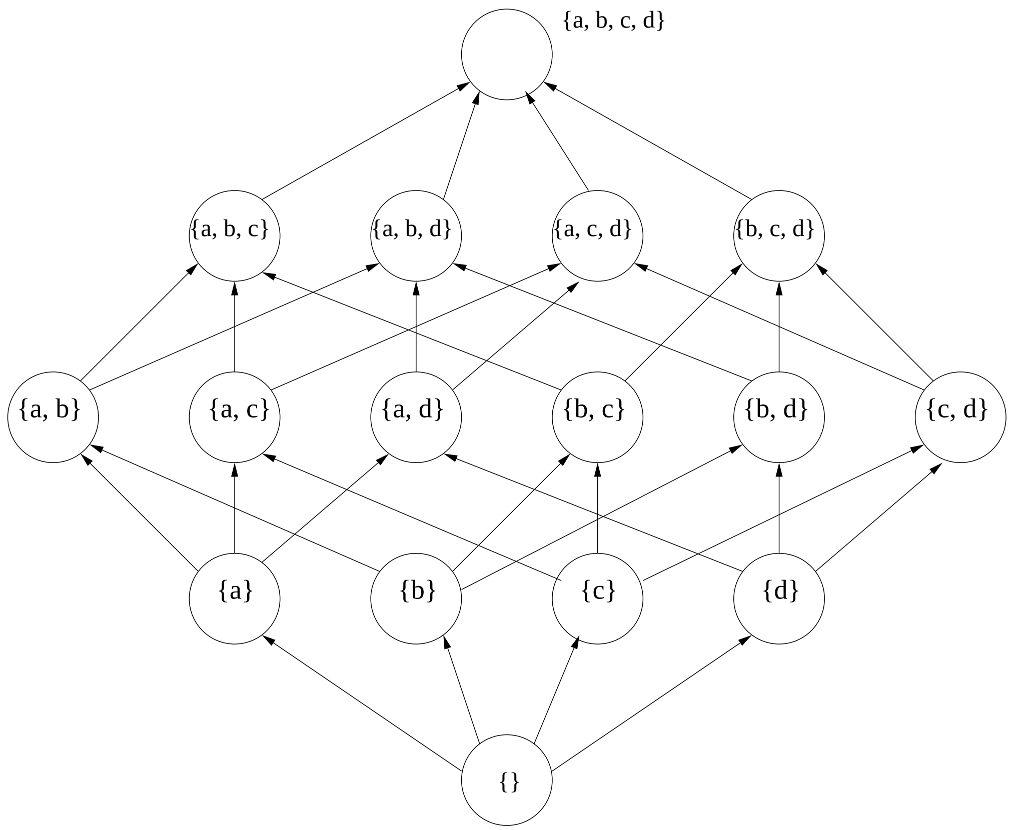
Problem 6.2

For each of the following posets, draw a Hasse diagram showing the partial order relation.

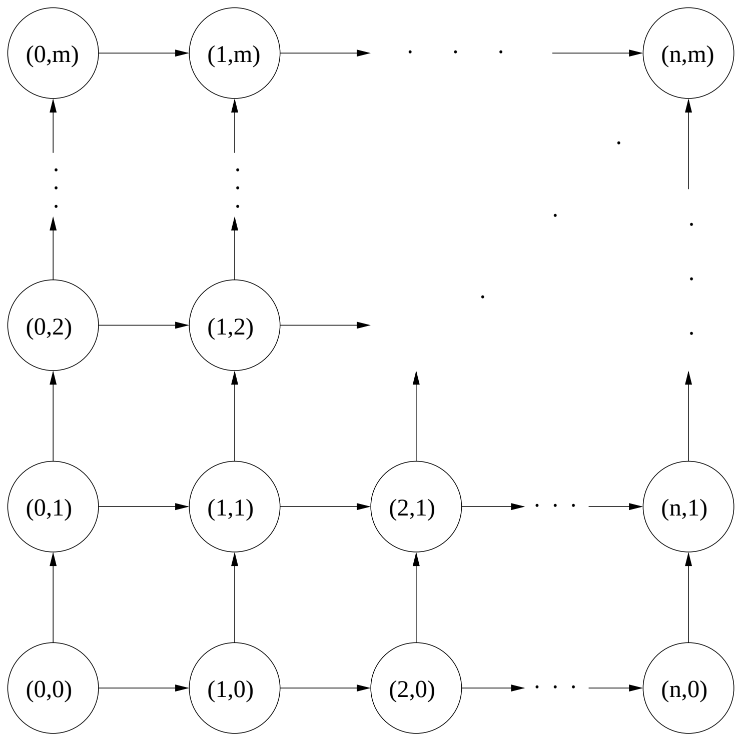
a) *The Power Set of a, b, c, d, where for any subsets A, B of a, b, c, d, $A \preceq B \Leftrightarrow A \subseteq B$. (This is also called the* **inclusion order**.*)*

b) *Given the totally ordered sets* $0, \ldots, n$ *and* $0, \ldots, m$ *with the ordinary* $\leq$ *orders, consider the Cartesian product of these sets, with* $(a, b) \preceq (c, d) \Leftrightarrow (a \leq c) \wedge (b \leq d)$.

c) *The set of finite sequences (strings) of the letters* a, b*. For example, abbab is such a sequence, as is* ϵ*, the empty sequence.* $S_1 \preceq S_2$ *iff* S_1 *is an* initial subsequence *of* S_2 *(also called the* prefix *of* S_2*); in other words, if* n *is the number of letters in* S_1*, then the first* n *letters of* S_2 *are exactly the sequence* S_1.



a)


(0,m)
(1,m)
(n,m)
(0,2)
(1,2)
(0,1)
(1,1)
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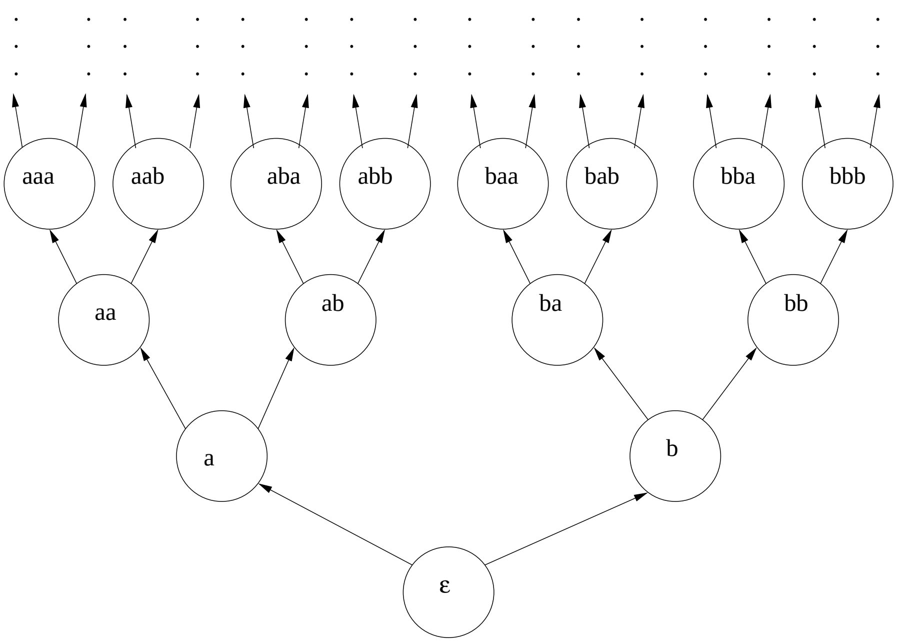
(0,0)
(1,0)
(2,0)
(n,0)

b)


aaa
aab
aba
abb
baa
bab
bba
bbb
aa
ab
ba
bb
a
b
ε

c)

Problem 6.3

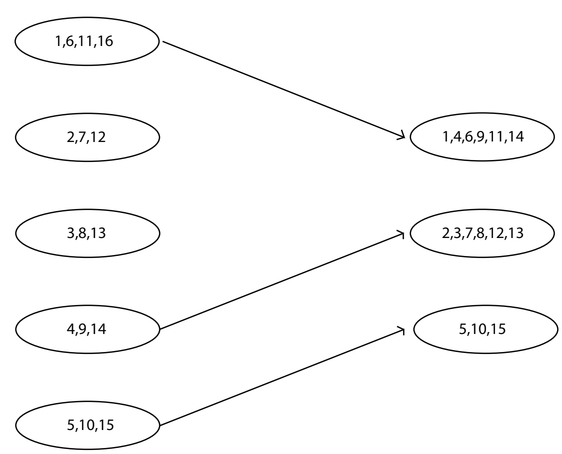
For each of the following relations, prove whether it is an equivalence relation. If so, list the equivalence classes. Otherwise, display the relation graphically.

a. $R = \{(a, b) \mid a = b^2 \; mod \; 5\}$ *on set* $\{1, 2, 3...16\}$

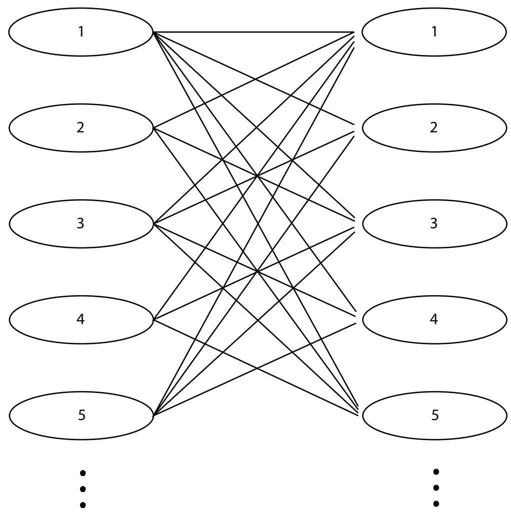
b. $R = \{(a, b) \mid gcd(a, b) = 1\}$ *(i.e., a and b are prime, having no common divisors except for 1.) for* $\{1, 2, ...10\}$

c. $R = \{(a, b) \mid a = (2b)^2\}$ *for* $\{-5, -4, -3...5\}$

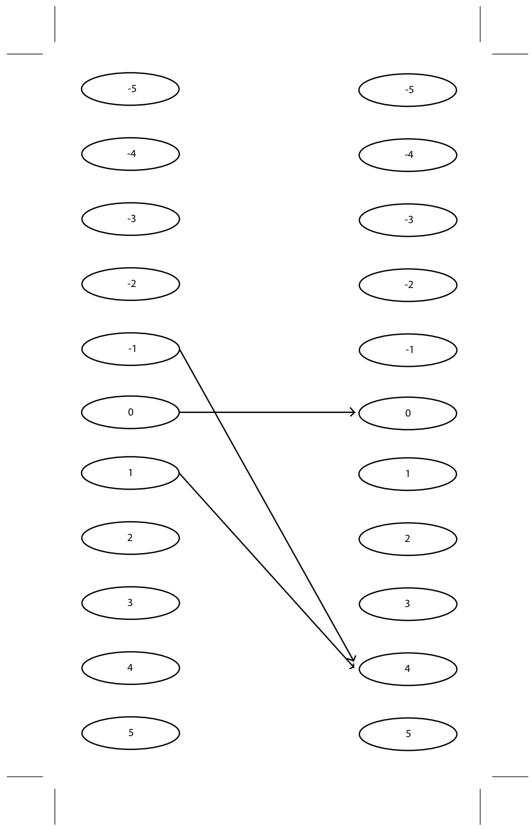
a. This relation is not reflexive. For example $2 \neq 2^2 mod 5$. Because it is not reflexive it cannot be an equivalence relation.



b. This relation is not reflexive. For example $gcd(2, 2) = 2 \neq 1$. Because it is not reflexive it cannot be an equivalence relation.



c. This relation is not reflexive. For example $-5 \neq (-10)^2$. Because it is not reflexive it cannot be an equivalence relation.



Problem 6.4

Let C be the set of the 16 2x2 chessboards whose squares are all colored either red or blue. Let R be a relation on C such that for two such chessboards, c_1 and c_2, $(c_1, c_2) \in R$ if and only if c_2 can be obtained from c_1 by a (finite) sequence of 90 degree rotations. (Imagine a nail in the center of the board, where the four squares meet. "One rotation," then, is accomplished by rotating the board $\pi/2$ or $-\pi/2$ about the nail.)

a. *Show that R is an equivalence relation.*

b. *What are the equivalence classes of R?*

a. If r is a rotation, the r^-1 denote the rotation in the opposite direction.

- *Reflexive* For any $c \in C$, cRc since C can be obtained from itself by a sequence of 0 rotations.
- *Symmetric* Let $c_1, c_2 \in C$ such that $c_1 R c_2$. This means there is a sequence of rotations $r_1, \ldots, r_n$ that transforms c_1 into c_2. This means that the sequence $r_n^{-1}, \ldots, r_1^{-1}$ transforms c_2 into c_1, so $c_2 R c_1$.
- *Transitive* Let $c_1, c_2, c_3 \in C$ such that $c_1 R c_2$ and $c_2 R c_3$. This means there is a sequence of rotations $r_1, \ldots, r_n$ that transforms c_1 into c_2, and a sequence $s_1, \ldots, s_n$ that transforms c_2 into c_3. Then the combined sequence $r_1, \ldots, r_n, s_1, \ldots, s_n$ will transform c_1 into c_3, so $c_1 R c_3$.

b. We denote a board by a string of length four, with r and b denoting red and blue, respectively. The first letter represents the upper left square, and the following letters continue in clockwise order. The equivalence classes are: $\{rrrr\}$, $\{bbbb\}$, $\{rrbr, rrrb, rbrr, brrr\}$, $\{bbrb, bbbr, brbb, rbbb\}$, $\{rbrb, brbr\}$, $\{rrbb, brrb, bbrr, rbbr\}$.

Problem 6.5

a. Let (S, R) be a poset. Show that (S, R^{-1}) is also a poset, where R^{-1} is the inverse relation of R. The poset (S, R^{-1}) is called the dual *of (S, R).*

b. Suppose that $(S, \preceq_1)$ and $(T, \preceq_2)$ are posets. If $\preceq$ is defined by $(s,t) \preceq (u,v)$ if and only if $s \preceq_1 u$ and $t \preceq_2 v$ for $s, u \in S$, and $t, v \in T$, show that $(S \times T, \preceq)$ is a poset

a. We must show that R^{-1} is a partial order.
Let $t, u, v \in R$.

Reflexive tRt, since R is a partial order. Thus $tR^{-1}t$ by definition of R^{-1}.

Antisymmetric Suppose $tR^{-1}u$ and $uR^{-1}t$. Then uRt and tRu since R is the inverse of R^{-1}. But R is a partial order, so $t = u$.

Transitive Suppose $tR^{-1}u$ and $uR^{-1}v$. Then uRt and vRu. Since R is a partial order, this implies vRt. But R^{-1} is the inverse of R, so $tR^{-1}v$.

b. We must show that $\preceq$ is a partial order.

Reflexive $\preceq_1$ and $\preceq_2$ are partial orders, so $s \preceq_1 s$ and $t \preceq_2 t$. Thus $(s,t) \preceq (s,t)$.

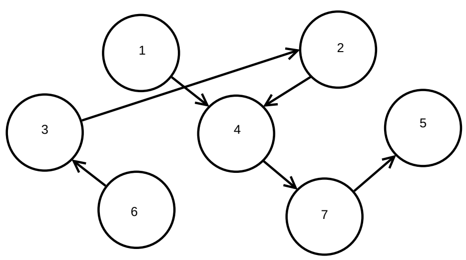
Antisymmetric Suppose $(s,t) \preceq (u,v)$ and $(u,v) \preceq (s,t)$. Then $s \preceq_1 u \preceq_1 s$ and $t \preceq_2 v \preceq_2 t$. Hence $s = u$ and $t = v$.

Transitive Suppose $(s,t) \preceq (u,v) \preceq (w,x)$. Then $s \preceq_1 u$, $u \preceq_1 w$, $t \preceq_2 v$ and $v \preceq_2 x$. By transitivity, $s \preceq_1 w$ and $t \preceq_2 x$. Hence $(s,t) \preceq (w,x)$.

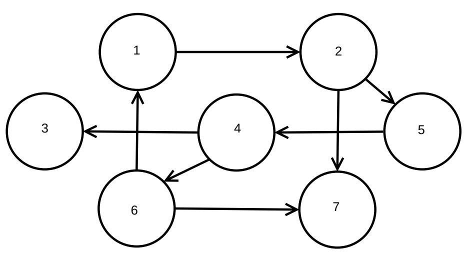
Problem 6.6

Solve for each of the following:

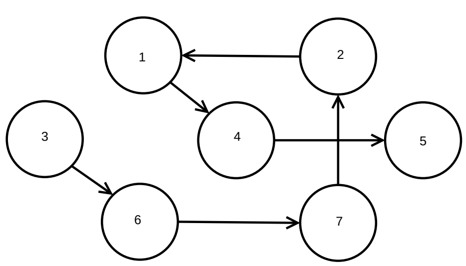
- *Can the graph be sorted topologically?*
- *If so, does there exist a unique sort?*
- *If there is more than one sort, enumerate all possible sorts.*



a.



b.



c.

a. There are 4 topological sorts for this graph: 1,6,3,2,4,7,5; 6,1,3,2,4,7,5; 6,3,1,2,4,7,5; and 6,3,2,1,4,7,5.

b. There is a cycle in this graph so there is no topological sort.

c. There are 6 topological sorts for this graph: 3,4,5,6,7,2,1; 3,4,6,5,7,2,1; 3,4,6,7,5,2,1; 3,6,7,4,5,2,1; 3,6,4,5,7,2,1; 3,6,4,7,5,2,1.