Homework 7

Solution Key

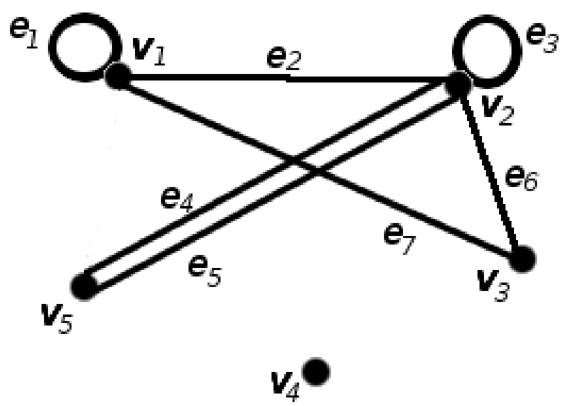
All homeworks are due at 1:00pm in the CS22 bin on the CIT second floor, opposite the elevators.

Write your full name *and the problem number on each piece of paper you hand in and then staple.*

Reading*: Chapter 11: 11.1-11.6 (up to p. 728); Chapter 7, Section 7.1-7.2 (up to p. 406)*

Problem 7.1

In the following graph, vertices are denoted by v's and edges by e's. Do the following:



a) *Find all edges that are incident on* v_1

b) *Find all vertices that are adjacent to* v_3

c) *Find all edges that are adjacent to* e_1

d) *Find all loops*

e) *Find all parallel edges*

f) *Find all isolated vertices*

g) Find the degree of v_3

h) Find the total degree of the graph

a) e_1, e_2, e_7

b) v_1, v_2

c) e_2, e_7

d) Self-loops: e_1, e_3 and cycle: $\{e_2, e_6, e_7\}$

e) e_4 and e_5

f) v_4

g) 2

h) 14

Problem 7.2

For each of a-d), either draw a graph with the specified properties or explain why no such graph exists:

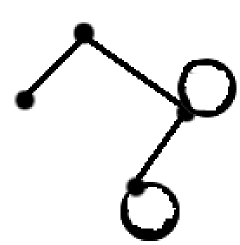
a) Graph with four vertices of degrees 1, 1, 1, and 4.

b) Graph with four vertices of degrees 1, 2, 3, and 4.

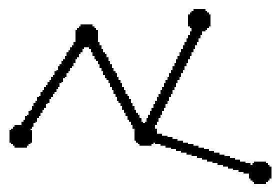
c) Simple graph with five vertices of degrees 1, 1, 1, 2, and 3.

d) Simple graph with four vertices of degrees 1, 2, 3, and 4.

a) No such graph exists. For any graph, the sum of the degrees of the vertices must be even (by the handshake theorem), but this graph would have the degrees sum to 7, which is odd.



b) Exists



c) Exists

d) No such graph exists. Suppose not, suppose there were a simple graph with four verticies of degrees 1,2,3 and 4. Then the vertex degree 4 would have to be connected by edges to four distinct vertices other than itself because of the assumption that the graph is simple (an hene has no loops or parallel edges). This contradicts the assumption that the graph has four vertices total. Hence there is no simple graph with four verticies of degrees 1,2,3, and 4.

Problem 7.3

Let G be a connected graph and let C be a circuit in G. Let G' be the subgraph obtained by removing all the edges of C from G and also any vertices that become isolated when the edges of C are removed. Prove that if G' is nonempty, then there exists a vertex v such that v is in both C and G'. The following lemma will be helpful: If a graph G is connected, then any two distinct vertices of G can be connected by a simple path.

Let G be a connected graph and let C be a circuit in G. Let G' be the nonempty subgraph obtained by removing all the edges of C from G and also any verticies that become isolated when the edges of C are removed. (We must show that there exists a vertex v such that v is in both C and G'). Pick any vertex v of C and any vertex w of G'. Since G is connected, there is a simple path from v to w (by the hint): v (in C) $= v_0e_1v_1e_2v_2...v_{i-1}e_iv_i$ (in C) $e_{i+1}v_{i+1}$ (not in C) $...v_{n-1}e_nv_n = w$. Let i be the largest subscript such that v_i is in C. If $i = n$, then $v_n = w$ is in C and also in G' and we are done. If $i < n$, then v_i is in C and v_{i+1} is not in C. This implies that e_{1+1} is not in C (for if it were, both endpoints would be in C by the definition of a circuit). Hence when G' is formed by removed all the edges and resulting isolated verticies form G, then e_{i+1} is not removed. That means that v_i does not become an isolated vertex, so v_i is not removed from either. Hence v_i is in G'. Consequently, v_i is in both C and G' (as was to be shown).

Problem 7.4

Prove that given any two distinct vertices of a tree, there exists a unique path from one to the other.

Proof by contradiction. Suppose not. Suppose that for some tree T, u and v are distinct verticies of T, and P_1 and P_2 are distinct paths joining u and v. Let P_1 be denoted $u = v_0, v_1, v_2, ..., v_m = v$, and let P_2 be denoted $u = w_0, w_1, w_2, ..., w_n = v$. Because P_1 and P_2 are distinct, and T has no parallel edges, the sequence of vertices in P_1 must diverge from the sequence of vertices in P_2 at some point. Let i be the least integer such that $v_i \neq w_i$. Then $v_{i-1} = w_{i-1}$. Let j and k be the least integers greater than i so that $v_j = w_k$, which must exist since both paths must converge back on the vertex $v_m = w_m = v$. Then:

$$v_{i-1}v_iv_{i+1}...v_j(= w_k)w_{k-1}...w_iw_{i-1}(= v_{i-1})$$

is a circuit in T. The existence of such a circuit contradicts the fact that T is a tree. Hence the supposition must be false. That is, given any tree with vertices u and v, there is a unique path joining u and v.

Problem 7.5

Let $G = (V, E)$ be a simple graph. Let R be the relation on V consisting of pairs of vertices (u, v) such that there is a path from u to v or such that $u = v$. Show that R is an equivalence relation.

R is reflexive by definition.

Assume that $(u, v) \in R$; then there is a path from u to v. Then $(v, u) \in R$ since there is a path from v to u, namely, the path from u to v traversed backward.

Assume that $(u, v) \in R$ and $(v, w) \in R$; then there are paths from u to v and from v to w. Putting these two paths together gives a path from u to w. Hence $(u, w) \in R$. It follows that R is transitive.

Problem 7.6

The Brown CS department has decided to secede from Facebook, creating their own internal social network called CodeBook. *For space considerations,* *the programmers need to know how many possible friendships will exist in this new network. Determine the* m*aximum total number of friendships that can exist in* C*odeBook if there are n people in the network, and prove your solution.*

We represent each individual as a vertex on an undirected graph G, and each friendship as an edge between two vertices. For the maximum number of friendships, there will be an edge connecting all vertices. The total number of edges is the total number of friendships.

If we have a graph of n vertices/people, and a new person enters the network who 'friends' everyone in the network, that will add 1 vertex and n edges to G. The next person would add $n+1$ edges/friendships. This is a simple sarithmetic summation:

$$\text{The maximum number of friendships} = \tfrac{n(n-1)}{2}$$

Proof (By Induction): Let $P(n)$ be the predicate the above foruma holds for n people in the network.

Base Case: $n = 1$

$$\frac{n(n-1)}{2} = \frac{1 * 0}{2} = 0$$

which makes sense since a network consisting of one person will have zero friendships.

Inductive Hypothesis: For $n \geq 1$, assume $\frac{n(n-1)}{2}$ is the maximum number of friendships for a network of n people.

We must prove that $\frac{n(n+1)}{2}$ is the maximum number of friendships for a network of $n+1$ people.

The new person must shake hands with n people already in the network. Thus:

$$
\begin{aligned}
\frac{n*(n-1)}{2}+n &= \frac{n*(n-1)}{2}+\frac{2n}{2}\\
&= \frac{n*(n-1)+2n}{2}\\
&= \frac{n*(n-1+2)}{2}\\
&= \frac{n*(n+1)}{2}
\end{aligned}
$$

Thus, $\forall n P(n) \Rightarrow P(n+1)$ and completes the proof.

Problem 7.7

Show that in any simple graph $G = (V, E)$ there is a path from any vertex v of odd degree to some other vertex of odd degree.

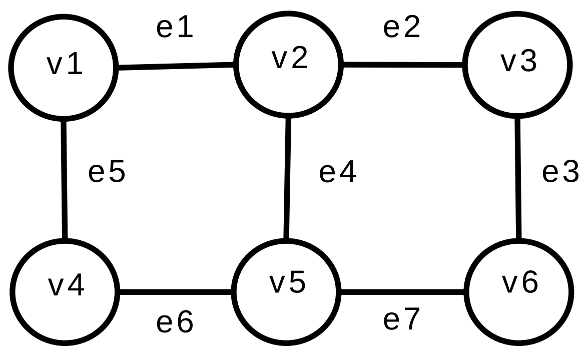
Consider the set $V' \subseteq V$ of vertices that are connected to v and $E' \subseteq E$ the set of edges connecting them. $G' = (V', E')$ is then a subgraph of G that is not connected in G to any vertex $x \notin V'$. The total degree of G' must be even (see Corollary 11.1.2, Epp p.660). Suppose that for all $w \in V'$, if $w \neq v$ then the degree of w is even. If $V' = \{w_1, w_2, ..., w_m, v\}$, then the total degree of the graph is as follows:

$$
\begin{aligned}
deg(G') &= deg(w_1) + \cdots + deg(w_m) + deg(v)\\
&= (2k_1) + \cdots + (2k_m) + (2k_0 + 1), \quad k_i \in \mathbb{Z}\ \forall i\\
&= 2(k_1 + \cdots + k_m + k_0) + 1\\
&= 2k + 1, \quad k \in \mathbb{Z}
\end{aligned}
$$

We have then shown the total degree of G' to be odd, which contradicts our knowledge that the total degree of the graph must be even. Then it is not possible that all vertices other than v have even degree. Therefore, there exists at least one vertex $v' \in V'$ such that $v' \neq v$ and the degree of v' is odd.Since G' was defined to be connected, there must exist a path from v to v' in G'. We have therefore proved our claim that in a simple graph G, there exists a path from any vertex v of odd degree to another vertex of odd degree.

Problem 7.8

Consider the graph G shown below:



How many spanning trees does it have?

15 –solution–