

# Homework 8

## Solution Key

All homeworks are due at 1:00pm in the CS22 bin on the CIT second floor, opposite the elevators.

Write your full name and the problem number on each piece of paper you hand in and then staple.

**Reading:** Chapter 6: Section 6.1 and 6.2. Chapter 7: Section 7.2 (pp. 415-417), 7.3, 7.4, 7.5.

### Problem 8.1

For  $\Sigma = \{0, 1\}$ , draw finite state machines that accept:

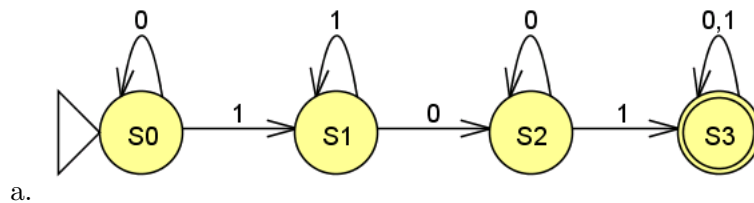
- a. All strings that contain the subsequence 101.

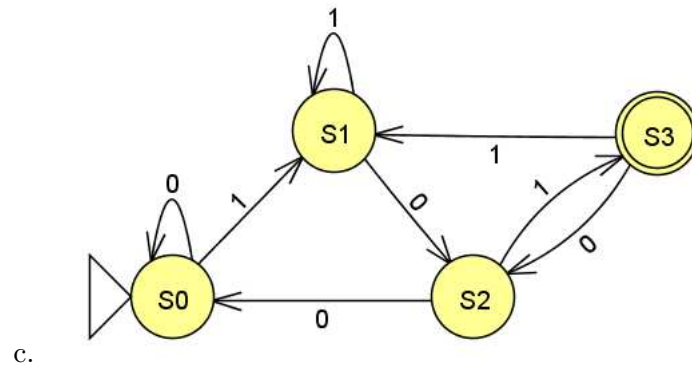
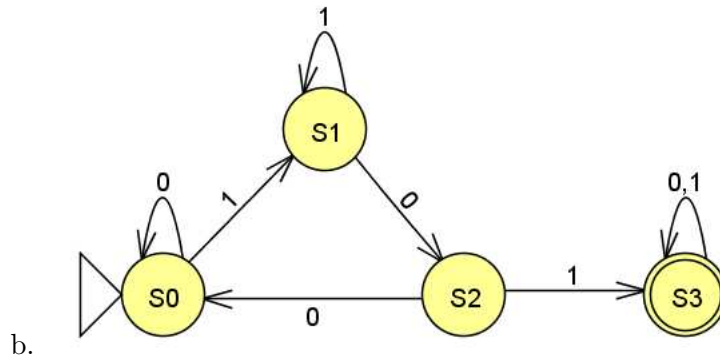
Note: A subsequence is an ordered selection of characters from a string. For example, the strings 1001 and 000100010 contain the sequence 101 as a subsequence.

- b. All strings that contain the substring 101.

Note: Characters in substrings must be consecutive. So the strings 1001 and 000100010 do NOT contain the substring 101, but 0110010100 does.

- c. All strings that end with 101.



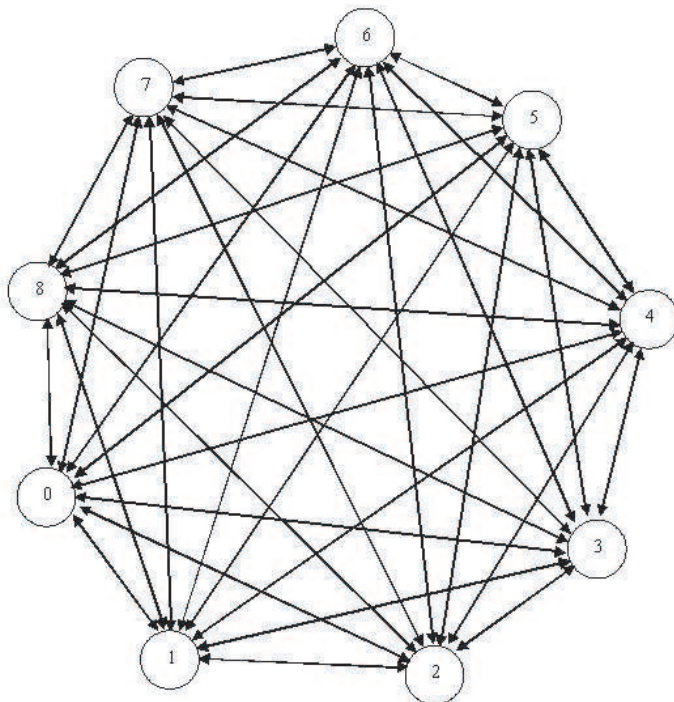


**Problem 8.2**

Construct the state diagram of a finite state machine that when given the sequence of the digits of a number (in base 10) from right to left, computes the remainder of its division by 9. Will the machine still work if given the digits of the number in the opposite order? Be sure to justify your answer and briefly explain how your machine works.

**SOLUTION:**

The state diagram is given below:



It is based on the principle that the remainder of a number's division by nine is equal to the remainder of division of the sum of its digits by nine (proved in an earlier homework). There are nine states, each corresponding to a possible remainder of a division by 9. We begin in state 0. The inputs are the digits of the number (which can be given in any order, since addition of numbers is commutative). The transitions are not labelled for the sake of clarity, but they follow the following formula: the input is added to the value of the state and the result modulo nine is the next state. Note that included in this state machine (but not in the diagram) are self loops for an input of zero or nine. Note also that there must be an end of number signal which indicates that all the digits of the number have been entered.

### Problem 8.3

*Explain why  $f$  is one-to-one (injection), onto (surjection), or a bijection. Show EACH attribute by proof or counterexample where appropriate.*

a.  $f: \mathbb{R} \rightarrow \mathbb{R}: f(x) = e^x$

b.  $f: \mathbb{R} \rightarrow \mathbb{R}^+: f(x) = x^4$

c.  $f: \mathbb{N} \rightarrow \mathbb{N}: f(x) = 2x$

d.  $f: \mathbb{R} \rightarrow \mathbb{Z}: f(x) = \lceil x \rceil$

When giving counterexamples, any similar counterexample is fine. Also, similar arguments will be accepted for explanations of true statements.

a.  $f: \mathbb{R} \rightarrow \mathbb{R}: f(x) = e^x$

- **One-to-one.** Consider  $x_1, x_2 \in \mathbb{R}$  such that  $f(x_1) = f(x_2)$ . Then  $e^{x_1} = e^{x_2}$  and  $\ln(e^{x_1}) = \ln(e^{x_2})$ . Because  $\ln x$  is unique for all  $x \in \mathbb{R}$ , we can then say that  $x_1 = x_2$ .
- **Not Onto.** Consider -1 as an element of  $\mathbb{R}$ . No value of  $e^x$  maps to a negative number (and so is  $\ln -1$  undefined, so the entire codomain  $\mathbb{R}$  is not mapped to, and thus  $f$  is not onto).
- **Not A One-To-One Correspondence.** Since  $f$  is not onto,  $f$  cannot be a one-to-one correspondence.

b.  $f: \mathbb{R} \rightarrow \mathbb{R}^+: f(x) = x^4$

- **Not one-to-one.** Consider  $x_1 = -1$  and  $x_2 = 1$ .  $f(x_1) = (-1)^4 = f(x_2) = (1)^4 = 1$ , but  $x_1 \neq x_2$ , so  $f$  cannot be injective.
- **Onto.** Consider an arbitrary element  $y \in \mathbb{R}^+$ . Since  $y$  is a positive real number, all real powers of  $y$  are positive real numbers. In particular,  $y^{1/4}$  is a positive real number and is therefore a member of the domain  $\mathbb{R}$ .  $f(y^{1/4}) = (y^{1/4})^4 = y$ . Since for every element of the codomain there is an element of the domain which maps to it,  $f$  is surjective.
- **Not a One-To-One Correspondence.** Since  $f$  is not injective,  $f$  is not bijective.

c.  $f: \mathbb{N} \rightarrow \mathbb{N}: f(x) = 2x$

- **One-to-one.**  $2x$  is unique over  $\mathbb{N}$  - there exists no two distinct  $x$  such that  $\frac{2x}{2} = x = x$  except for  $x$  itself by the reflexive property of real numbers.

- **Not Onto.** Consider 3 as an element of  $\mathbb{N}$ .  $2x = 3, x = \frac{3}{2}$  which is not an element of the domain  $\mathbb{N}$ . Since  $\frac{3}{2}$  is in the codomain and not mapped to by the domain,  $f$  is not onto.
- **Not A One-To-One Correspondence.** Since  $f$  is not onto,  $f$  cannot be a one-to-one correspondence.

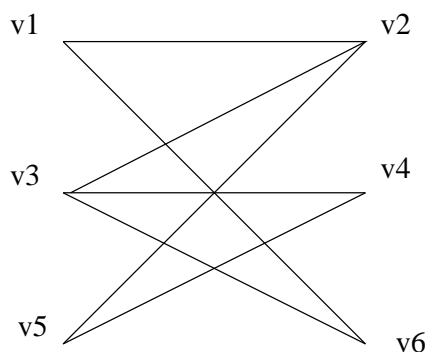
d.  $f: \mathbb{R} \rightarrow \mathbb{Z}: f(x) = \lceil x \rceil$ .

- **Not one-to-one.** Consider  $x_1 = .5$  and  $x_2 = .6$ .  $f(x_1) = \lceil .5 \rceil = 1$ ,  $f(x_2) = \lceil .6 \rceil = 1$ , but  $x_1 \neq x_2$ . Therefore  $f$  cannot be injective
- **Onto.** Consider an arbitrary element  $y \in \mathbb{Z}$ . Since every integer is a real number,  $\mathbb{Z} \subset \mathbb{R}$  and  $y$  is an element of both the domain and codomain. When  $y$  is an integer,  $f(y) = y$ , so for each element of the codomain there is an element of the domain that maps to it.  $f$  is therefore surjective.
- **Not a One-To-One Correspondence.** Since  $f$  is not injective,  $f$  cannot be bijective.

### Problem 8.4

A 2-colored graph  $G$  is a simple graph each vertex of which can be assigned one of two colors so that no edge connects two vertices of the same color.

For example, the graph below is 2-colored:



Prove that an undirected 2-colored graph cannot contain a cycle that has an odd number of vertices.

Proof: Assume that there exists a two-colored graph containing a cycle denoted by

$$a_1, b_1, a_2, b_2, \dots, a_n, b_n, a_1$$

$$\forall i \in N, i \leq n, a_i \in V_1, b_i \in V_2$$

where  $V_1$  is the group of vertices of one color and  $V_2$  is the group of vertices of the second color.

Note that we must alternate between vertices in  $V_1$  and  $V_2$  because the edges only connect vertices of opposite colors. The total number of vertices in this sequence is  $2n$  (since we do not count the first vertex twice). This is always an even number. Therefore a cycle in an 2-colored graph cannot contain an odd number of vertices.

### Problem 8.5

Let  $S = \{1, 2, \dots, 2n\}$  for some integer  $n$ . Show that for any  $T \subset S$  such that  $|T| = n + 1$ , there are elements  $x, y \in T$  such that  $x$  and  $y$  are relatively prime.

(Hint: Show that there are elements  $x, y \in T$  such that  $|x - y| = 1$ , then go from there.)

First, we will prove that  $T$  contains two elements  $x, y$  that are adjacent, that is,  $|x - y| = 1$ . Assume it does not. Then for any distinct  $x, y$ ,  $|x - y| \geq 2$ . But by the pigeonhole principle, any subset of  $S$  that satisfies this constraint has at most  $\frac{|S|}{2} = n$  elements. This contradicts the assumption that  $|T| = n + 1$ . Thus there is some pair  $x, y \in T$  such that  $|x - y| = 1$ .

Now, we show that  $x$  and  $y$  are relatively prime. Assume there is some  $d$  such that  $d|x$  and  $d|y$ . Then  $d|(x - y)$ , so  $d|1$ , so  $d = 1$ . Therefore  $x$  and  $y$  are relatively prime.

### Problem 8.6

Using composability of functions show:

a.  $f(y) = e^{3y^2+2} + 1$  is not injective from  $\mathbb{R}$  to  $\mathbb{R}$

b.  $f(x) = (2x + 4)^5$  is bijective from  $\mathbb{R}$  to  $\mathbb{R}$

- a. Injective means that  $f(x) = f(y)$  implies  $x = y$ . Suppose that  $f(y)$  is the composition  $f(y) = (g \circ h)(y)$  where  $g(y) = e^h(y) + 1$  and  $h(y) = 3y^2 + 2$ . In order for  $f(y)$  to be injective,  $h(y)$  must be injective from  $\mathbb{R}$  to  $\mathbb{R}$ . But this is not so.  $h(y) = 3y^2 + 2$  is not injective because not every member of the codomain has a unique map to the domain. Consider  $h(y) = 5$ .  $\sqrt{\frac{5-2}{3}} = \sqrt{1}$  has two solutions:  $-1$  and  $1$ , so both  $y = -1$  and  $y = 1$  map to  $h(y) = 5$ . By this counterexample,  $h(y)$  is not injective and therefore either is  $(g \circ h)(y)$ .
- b. Suppose that  $f(x)$  is the composition  $f(y) = (g \circ h)(x)$  where  $g(y) = h(y)^5$  and  $h(y) = 2x + 4$ . Bijective implies both surjective and injective:

Surjective proof:  $f(y) = (g \circ h)(x)$  is surjective if both  $g$  and  $h$  are surjective. Surjective means that  $\forall y \in Y$  (codomain),  $\exists x \in X$  (domain) such that  $g(x) = y$  and  $\forall v, \exists u$  such that  $h(v) = u$ . For  $h(v)$ , assume  $u = \frac{v-4}{2}$ . Therefore,  $2(\frac{v-4}{2}) + 4 = v$ . For  $g(x)$  assume  $x = \sqrt[5]{y}$ . Then  $g(x) = (\sqrt[5]{y})^5 = y$ . Therefore both  $g$  and  $h$  are surjective over  $\mathbb{R}$  to  $\mathbb{R}$  and so is  $f(x)$ .

Injective proof:  $f(y) = (g \circ h)(x)$  is injective if both  $g$  and  $h$  are injective. Injective means that  $g(x) = g(y)$  implies  $x = y$  and  $h(u) = h(v)$  implies  $u = v$ . For  $g$ , assume that  $g(x) = g(y)$ . Therefore,  $x^5 = y^5$ . Taking the 5th root, which only has one unique solution,  $x = y$ . Therefore,  $g$  is injective. For  $h$ , assume that  $h(u) = h(v)$ . Therefore,  $2u + 4 = 2v + 4$ . Subtracting 4 and solving for  $u$ , we see that  $u = v$ . Therefore  $h$  is also injective and the composition  $(g \circ h)(x)$  is injective.