Homework 9

Solution Key

All homeworks are due at 1:00pm in the CS22 bin on the CIT second floor, opposite the elevators.

Write your full name *and the problem number on each piece of paper you hand in and then staple.*

Reading*: Chapter 6: 6.3, 6.4, 6.5, 6.6, 6.7.*

Problem 9.1

The CS22 TAs meet every Wednesday for a staff meeting at a circular table. Each meeting is attended by 5 male individuals (Alex, Ashwin, Ben, Chris, and Tom) and three women (Nell, Catherine, and Thea).

a. *Alex decided that he wants to sit between two women. He removed one of the chairs so now the round table has 7 chairs. The women arrived first and, as before, sat next to each other. Then the guys arrived and occupied the four remaining seats. Finally, Alex arrived and brought an extra chair, which he placed so as to sit between two women. How many possible arrangements of the TAs at the round table are there in this case? ALSO, for the general case, what if we have w women and m men (including Alex)?*

   There are 3! ways in which the women can sit and before Alex arrives, 4! ways the men can sit. Once Alex comes, he can sit in one of 2 different places thus $3! \cdot 4! \cdot 2 = 288$. For the general case $w! \cdot (m-1)! \cdot (w-1)$.

b. *In how many ways can you sit the men and women around the circular table so that no two women sit next to each other?*

   There are two possible configurations in which two women are not sitting together (a group of three men or two groups of two men). Note that these two configurations are rotationally asymmetric, so you do not need to account for rotationally equivalent solutions. For each configuration, we can order the 5 men in 5! ways and the women in 3! ways. Thus, the total number of configurations in which no two women sit next to each other is $2 \cdot 3! \cdot 5! = 1440$.

c. *Nell decided to treat the TAs to chocolate. There are five flavored chocolates (Amaretto, Hazelnut, Almond, Fish, and Orange) and three plain chocolates (White, Dark and Milk). The TAs will have one chocolate each. The men will have the flavored chocolates, while the women will have the plain chocolates. How many ways are there to distribute chocolates to TAs? What if there are w women, m men, $w + m$ distinct chocolates, w of which are meant for women, and the remaining m meant for men? Is there a bijection to the set of arrangements for part (a)?*

   The women can choose chocolats in 3! ways and the men can choose them in 5! ways, $3! \cdot 5! = 720$. For the general case we have $w! \cdot m!$.

   There is not a bijection.

d. *The next time around, Nell left eight Milk chocolates for the TAs to eat during the grading session. The men arrived first and decided that (1) each man should have at least one chocolate; and (2) all chocolate should be distributed to men. How many ways are there to distribute these chocolates to the TAs, if the men get their way? (Note that all chocolates are the same, the only thing that matters is how many chocolates each TA got to eat.) What if we have m men and $c \geq m$ pieces of chocolate?*

   We begin by giving one chocolate to each man, this leaves 3 which we can distribute using a balls and bins mechanism which gives 7 choose $3 = 35$ ways. In general, we have $c - m$ chocolates to distribute to m men, so the stars and bars formula gives $\binom{(c-m)+m-1}{c-m} = \binom{c-1}{c-m}$.

Problem 9.2

Consider a standard deck of 52 cards. How many 5-card sets contain

a. *the queen of hearts?*

b. *no cards with values lower than 5 (aces have value 1)?*

c. *exactly 1 pair?*

d. *at least 3 diamonds?*

a. Every hand containing the queen of hearts must contain that card and any four of the remaining 51 cards. The total number of hands is then

$$\binom{51}{4} = 249900$$

b. The set of cards with values greater than or equal to 5 is $(13-4)*4 = 36$. The number of sets with no cards of value less than 5 is then

$$\binom{36}{5} = 376992$$

c. If a hand contains exactly one pair, it must contain two cards in one denomination and one card in each of three other denominations. There are 13 ways to choose the denomination for the pair and $\binom{4}{2}$ ways to choose two cards (two suits out of 4) from that denomination. There are $\binom{12}{3}$ ways to choose the denominations for the remaining denominations, and 4 ways to choose a card for each denomination. The total number of hands is then

$$13 * \binom{4}{2} * \binom{12}{3} * 4^3 = 1098240$$

d. The set of diamonds contains 13 cards, and the set of non-diamonds contains 39 cards. A hand that contains at least 3 diamonds must contain 3 diamonds and 2 non-diamonds, 4 diamonds and 1 non-diamond, or 5 diamonds. The total number of hands is then

$$\binom{13}{3} * \binom{39}{2} + \binom{13}{4} * 39 + \binom{13}{5}$$

Problem 9.3

Give the coefficient of the given term in the expanded expression

a. x^4y^7 *in* $(3x^2 + 5y)^9$

b. $p^{12}q^{44}$ *in* $(7p^3 + 2q^4)^{15}$

a. The coefficient of x^4y^7 is $3^2 5^7 \binom{9}{2} = 25312500$.

b. The coefficient of $p^{12}q^{44}$ is $7^4 2^{11} \binom{15}{4} = 6712043520$

Problem 9.4

How many integer solutions are there to the system

$$
\begin{gathered}
x_1 + x_2 + x_3 + x_4 = 40 \\
1 \leq x_1 \leq 5 \\
2 \leq x_2 \leq 7 \\
3 \leq x_3 \leq 9 \\
5 \leq x_4
\end{gathered}
$$

There are two ways to solve this problem.

The quick way:

We can redifine the variables such that:

$$
\begin{gathered}
y_1 = x_1 - 1 \\
y_2 = x_2 - 2 \\
y_3 = x_3 - 3 \\
y_4 = x_4 - 5
\end{gathered}
$$

and:

$$
\begin{gathered}
0 \leq y_1 \leq 4 \\
0 \leq y_2 \leq 5 \\
0 \leq y_3 \leq 6 \\
0 \leq y_4
\end{gathered}
$$

$$y_1 + y_2 + y_3 + y_4 = 29$$

Notice that y_4 has no constraints, since given the maximum values for y_1, y_2, and y_3, $y_4 \geq 14$. Thus, we can set y_1, y_2, and y_3 to any allowed value, and y_4 can take on any value necessary so that the sum equals 40.

There are 5 possible values for y_1, 6 possible values for y_2, and 7 possible values for y_3.

Thus, there are $5 * 6 * 7 = 210$ possible solutions.

The second method:

We can represent the four values of x as 4 boxes, filling up with 40 $\times$'s. We need to assign 11 of these $\times$'s in order to fulfill minimum requirements:

$$\times | \times \times | \times \times \times | \times \times \times \times \times$$

Thus, there are 29 $\times$'s left for 4 boxes, where repitition is allowed. We can use the general formula:

$$\binom{r+n-1}{r}$$

where n is the number of boxes and r is the number of items to put in the boxes. This gives us:

$$\binom{29+4-1}{29} = \binom{32}{29}$$

combinations. However, we need to take away cases where we've gone over the maximum. There are three cases where we've gone over the maximum for exactly one box. For box 1, we would assign 6 $\times$'s to the box initially (along with the minimum requirements for the others), leaving 24 $\times$'s the 4 boxes (since we could also get penalties by filling up box 1 with more than 6). For all three boxes with maximum constraints, this comes to:

$$\binom{24+3}{24} + \binom{23+3}{23} + \binom{22+3}{22}$$

However, we've double counted here. We need to take into account situations where we've gone over the maximum on two boxes. For example, we could fill box 1 and 2 with 12 total $\times$'s, along with the minimum 3 for box 3 and 5 for box 4. By extending this logic to all cases of two penalties, this comes to:

$$\binom{18+3}{18} + \binom{17+3}{17} + \binom{16+3}{16}$$

Finally, we need to add back the case where we've gone over the maximum in all 3 boxes with constraints. This comes to:

$$\binom{11+3}{11}$$

The final answer is:

Combinations - (single penalties) + (double penalties) - (triple penalties), or:

$$\binom{32}{29} - \left(\binom{27}{24} + \binom{26}{23} + \binom{25}{22}\right) + \left(\binom{21}{18} + \binom{20}{17} + \binom{19}{16}\right) - \binom{14}{11}$$

This totals 210 possibilities.

Note: you should know both solutions, as problems on the final might not have the freedom that allowed the shortcut of the first solution!

Problem 9.5

Seven people enter an elevator in the basement. Each exists at floor 1,2,3, *or* 4. *In how many ways can this happen? Explain.*

Extra Credit

Problem 9.6

Use the inclusion-exclusion principle to find the number of primes not exceeding 100.

Recall that a composite integer is divisible by a prime not exceeding its square root. So, to find the number of primes not exceeding 100, first note that composite integers not exceeding 100 must have a prime factor not exceeding 10. Because the only primes less than 10 are 2, 3, 5, and 7, the primes not exceeding 100 are these four primes and those positive integers greater than 1 and not exceeding 100 that are divisible by none of 2, 3, 5, or 7. To apply the principle of inclusion-exclusion, let P_1 be the property that an integer is divisible by 2, let P_2 be the property that an integer is divisible by 3, let P_3 be the property that an integer is divisible by 5, and let P_4 be the property that an iteger is divisble by 7. Thus, the number of primes not exceeding 100 is

$$4 + N(P_1'P_2'P_3'P_4')$$

Since there are 99 positive integers greater than 1 and not exceeding 100, the principle of inclusion-exclusion shows that

$$N(P_1'P_2'P_3'P_4') = 99 - N(P_1) - N(P_2) - N(P_3) - N(P_4) + N(P_1P_2) + N(P_1P_3)$$

$$+N(P_1P_4) + N(P_2P_3) + N(P_2P_4) + N(P_3P_4) - N(P_1P_2P_3) - N(P_1P_2P_4)$$

$$-N(P_1P_3P_4) - N(P_2P_3P_4) + N(P_1P_2P_3P_4)$$

The number of integers not exceeding 100 (and greater than 1) that are divisible by all the primes in a subset of 2,3,5,7 is $\lfloor 100/N \rfloor$, where N is the product of the primes in this subset. (This follows since any two of these primes have no common factor.) Consequently,

$$N(P_1'P_2'P_3'P_4') = 99 - \lfloor \frac{100}{2} \rfloor - \lfloor \frac{100}{3} \rfloor - \lfloor \frac{100}{5} \rfloor - \lfloor \frac{100}{5} \rfloor + \lfloor \frac{100}{2*3} \rfloor + \lfloor \frac{100}{2*5} \rfloor + \lfloor \frac{100}{2*7} \rfloor$$

$$+ \lfloor \frac{100}{3*5} \rfloor + \lfloor \frac{100}{3*7} \rfloor + \lfloor \frac{100}{5*7} \rfloor - \lfloor \frac{100}{2*3*5} \rfloor - \lfloor \frac{100}{2*3*7} \rfloor - \lfloor \frac{100}{2*5*7} \rfloor - \lfloor \frac{100}{3*5*7} \rfloor$$

$$+ \lfloor \frac{100}{2*3*5*7} \rfloor$$

$$= 99 - 50 - 33 - 20 - 14 + 16 + 10 + 7 + 6 + 4 + 2 - 3 - 2 - 1 - 0 + 0$$

$$= 21$$

Hence, there are $4 + 21 = 25$ primes not exceeding 100.