CSCI 0220 **Discrete Structures and Probability** F. Preparata

Homework 10

Solution Key

All homeworks are due at 1:00pm in the CS22 bin on the CIT second floor, opposite the elevators.

Write your full name *and the problem number on each piece of paper you hand in and then staple.*

Reading*: Chapter 8: 8.1, 8.2, 8.3. Class notes chapters 1, 2.*

Note: *Please show your work and explain your reasoning on each problem in this assignment. We cannot award partial credit to students for incorrect answers that do not show their work, nor can we give full credit to students for correct answers that do not show their work.*

Problem 10.1

a. *A vampire arrives in Boston at day* 0 *and starts biting people at day* 1*. People bitten become vampires themselves. New vampires bite two persons on the next day after they were bitten and six persons every day afterwards. Write the recurrence relation of the number of vampires in* n *days for* $n \geq 2$*, assuming the first vampire is a newly made vampire.*

Let $V(n)$ denote the number of vampires on the nth day. We have the initial conditions $V(0) = 1$ and $V(1) = 3$.

The number of vampires on day $n+1$ is the number of vampires on day n plus all the people that were bitten on day n.

$$V(n+1) = V(n) + 2(V(n) - V(n-1)) + 6V(n-1) = 3V(n) + 4V(n-1)$$

b. *Solve the recurrence using the characteristic equation.*

Suppose the sequence $V(0)$, $V(1)$, $V(2)$, ... satisfies the recurrence ralation $V(n) = 3V(n-1) + 4V(n-2)$ for all integers $n \geq 2$, with initial conditions $V(0) = 1$ and $V(1) = 3$.
This sequence satisfies part of the hypothesis of the Distinct-Roots Theorem because it satisfies a second-order linear homogeneous recurrence relation with constant coefficients $A = 3$ and $B = 4$. The characteristic equation $t^2 - 3t - 4 = 0$ has two distinct roots r and s,

where $r = 4$ and $s = -1$, therefore the root condition is also satisfied. Therefore, by the Distinct-Roots Theorem the sequence $V(0)$, $V(1)$, ... satisfies the explicit formula:
$V(n) = Cr^n + Ds^n$,
where we obtain C and D by solving the system:
$V(0) = 1 = C.4^0 + D(-1)^0$
$V(1) = 3 = C.4^1 + D(-1)^1$
From the system we get $C = \frac{4}{5}$ and $D = -\frac{1}{5}$.
It follows that the sequence satisfies the explicit formula:
$V(n) = \frac{4}{5}.4^n + \frac{1}{5}(-1)^n$.

c. *Prove that your answer satisfies the recurrence relation.*

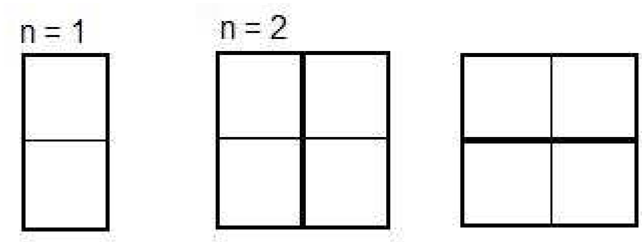
$$\begin{aligned}
V(n+1) &= \frac{4}{5}.4^{n+1} + \frac{1}{5}(-1)^{n+1} \\
3V(n) + 4V(n-1) &= 3.\frac{4}{5}.4^n + 3.\frac{1}{5}.(-1)^n + 4.\frac{4}{5}.4^{n-1} + 4.\frac{1}{5}.(-1)^{n-1} \\
&= 3.\frac{4}{5}.4^n + 4.\frac{4}{5}.4^{n-1} + \frac{1}{5}.(-1)^{n-1} \\
&= 3.\frac{4}{5}.4^n + \frac{4}{5}.4^n + \frac{1}{5}.(-1)^{n+1} \\
&= \frac{4}{5}.4^{n+1} + \frac{1}{5}.(-1)^{n+1} \\
&= V(n+1)
\end{aligned}$$

Problem 10.2

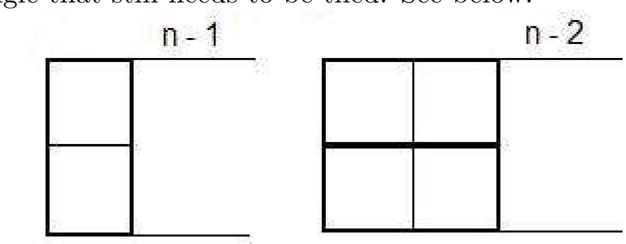
In how many ways can a 2 x n rectangle be tiled by 1 x 2 blocks? Prove your answer.

If F_n is the number of times a 2 x n rectangle can be tiled, $F_n = F_{n-1} + F_{n-2}$.

Proof: Base cases: $n = 1$, $n = 2$ $n = 1$ There is exactly one way to tile a 2 x 1 rectangle: 1 tile placed vertically, as shown below. $n = 2$ There are exactly two ways to tile a 2 x 2 rectangle: 2 tiles placed vertically, or 2 tiles placed horizontally.



Suppose we have a 2 x n rectangle. We know that the last block must be placed either horizontally or vertically. If the block is placed vertically, we know that there is a 2 x (n-1) rectangle that still needs to be tiled. If the block is placed horizontally, we know that another block must also be placed horizontally next to it. These two blocks cover a 2x2 rectangle, leaving a 2 x (n-2) rectangle that still needs to be tiled. See below.



There are therefore $F_{n-1}+F_{n-2}$ ways to tile a 2 x n rectangle. The recurrence is the same as for the Fibonacci numbers, but with initial conditions 1 and 2.

Now we need to solve the recurrence relation. The characteristic equation is $r^2 - r - 1 = 0$. We can solve this using the quadratic formula to get

$$r_1 = \frac{1+\sqrt{5}}{2},\ r_2 = \frac{1-\sqrt{5}}{2}$$

We then have

$$F_1 = c_1\left(\frac{1+\sqrt{5}}{2}\right) + c_2\left(\frac{1-\sqrt{5}}{2}\right) = 1$$

$$F_2 = c_1\left(\frac{1+\sqrt{5}}{2}\right)^2 + c_2\left(\frac{1-\sqrt{5}}{2}\right)^2 = 2$$

$$(1+\sqrt{5})c_1 + (1-\sqrt{5})c_2 = 2$$
$$(6+2\sqrt{5})c_1 + (6-2\sqrt{5})c_2 = 8$$
$$4c_1 + 4c_2 = 4 \Longrightarrow c_2 = 1 - c_1$$
$$(1+\sqrt{5})c_1 + (1-\sqrt{5})(1-c_1) = 2$$
$$2\sqrt{5}c_1 + 1 - \sqrt{5} = 2$$
$$c_1 = \frac{5+\sqrt{5}}{10}$$
$$c_2 = 1 - c_1 = \frac{5+\sqrt{5}}{10}$$
$$F_n = \left(\frac{5+\sqrt{5}}{10}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{5-\sqrt{5}}{10}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Problem 10.3

Suppose you have a basket containing 12 blue blocks and 6 white blocks. You draw 4 blocks from the basket without replacement. Assume all blocks are equally likely to be selected, and blocks of the same color are indistinguishable.

a. *List the simple events in the sample space. Order does not matter.*

b. *List the simple events for which there are exactly 2 blue blocks.*

Now suppose you add 2 yellow blocks and 3 red blocks to the 18 blocks already in the basket. Again, draw 4 blocks without replacement.

c. *What is the probability that all 4 blocks will be the same color?*

d. *What is the probability that all 4 blocks will be different colors, given that at least one block is yellow?*

a. $\{bbbb, bbbw, bbww, bwww, wwww\}$

b. $\{bbww\}$

c. The number of ways in which to choose 4 blocks from the 23 available is

$$\binom{23}{4} = 8855$$

The number of ways to choose 4 blocks of the same color is equal to the number of ways to choose 4 blue blocks plus the number of wayss to choose 4 white blocks (there are less than 4 yellow blocks and red blocks, so we need not consider those colors). This is

$$\binom{12}{4} + \binom{6}{4} = 510$$

The probability that all 4 blocks are the same color is then

$$\frac{510}{8855} = \frac{102}{1771} = .0576$$

d. Let us call the event that at least one yellow block is selected event A and the event that each block is a different color event B. We need to determine $\Pr(B|A)$. We know by the definition of conditional probability that

$$\Pr(B|A) = \frac{\Pr(A \cap B)}{\Pr(A)}$$

We must then determine the probabilities of $A \cap B$ and A occuring. We know that if 4 blocks of distinct colors are selected, at least 1 must be yellow. Then $\Pr(A \cap B) = \Pr(B)$. The number of ways 4 blocks of distinct colors can be selected is equal to the number of ways to select a blue block multiplied by the number of ways to select a white block, a yellow block, and a red block. This is

$$12 * 6 * 2 * 3 = 432$$

so

$$\Pr(B) = \frac{432}{8855}$$

. The number of ways to select at least 1 yellow block is equal to the number of ways to select 1 yellow block and 3 non-yellow blocks plus the number of ways to select 2 yellow blocks and 2 non-yellow blocks (there are only 2 available yellow blocks, and 21 available non-yellow blocks). This is

$$\binom{2}{1}\binom{21}{3} + \binom{2}{2}\binom{21}{2} = 2870$$

so

$$\Pr(A) = .3241$$

Using this information, we can determine that

$$\Pr(B|A) = \frac{432}{2870} = \frac{.0576}{.3241} = .1505$$

Problem 10.4

Are the following events independent?

a. *When rolling a die, that an even number shows up and that a number greater than three shows up.*

b. *When flipping a coin twice, that the first flip is a heads and that the two flips match.*

c. *When flipping a coin three times, the first coin flipping comes up tails and two (but not three) heads come up in a row.*

a. Let A be the event that an even number shows up and let B be the event that a number greater than three shows up.

$$P(A) \times P(B) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$
$$P(A \cap B) = \frac{1}{3}$$

The events are not independent.

b. Let A be the event that the first flip is heads and let B be the event that the two flips match.

$$P(A) \times P(B) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$
$$P(A \cap B) = \frac{1}{4}$$

The events are independent.

c. Let A be the event that the first flip is tails and let B be the event that two (but not three) heads come up in a row.

$$P(A) \times P(B) = \frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$$
$$P(A \cap B) = \frac{1}{8}$$

The events are independent.

Problem 10.5

You have a 100-inch glass ruler, with notches at each inch point from 0 to 100, inclusive. You drop it. It breaks at some location $k + \frac{1}{2}$, where k is an integer from 0 to 99. Each k is equally likely.

a. *What's the expected number of "inch marks" on the shorter piece?*

   To make this clearer, if we were talking about a 10-inch ruler and it broke at 7.5, there would be marks at 0, 1 2, 3, 4, 5, 6, and 7 on the longer piece, and marks at 8, 9, and 10 on the shorter piece, so the "number of marks" on the smaller piece would be three.

b. *What about the long end of the stick?*

a. The ruler is equally likely to break at any one of the hundred notches. Now, if it breaks at the 0.5 notch or the 99.5 notch, there will be one inch marker on the shorter edge. If it breaks at 1.5 or 98.5, there will be two inch markers on the shorter edge, and so on, up to breakage at the 49.5 notch or the 50.5 notch, in which case the short end of the stick will have 50 marks. In general, for all i less than 50, there is a $\frac{2}{100}$ probability that the shorter end will have i inch markers. So the expected value of inch markers on the shorter end of the ruler is

$$\sum_{i=1}^{50} \frac{2}{100} \cdot i$$

   This is just $\frac{1}{50} \sum_{i=1}^{50}$, which is $\frac{1}{50} \cdot \frac{50(51)}{2}$, which is $\frac{51}{2}$. So the expected number of inch markers on the short end of the stick is 25.5.

b. By the same argument as above, if the ruler breaks at 0.5 or 99.5, the long end of the stick will have 100 inch marks on it. If it breaks at 1.5 or 98.5, it will have 99 inch marks, and so on until it breaks at 49.5 or 50.5, in which case it will have 51 marks. So the expected number of marks on the long end of the stick is $\sum_{t=51}^{100} \frac{2}{100} \cdot i$. This is $\frac{1}{50} \sum_{t=51}^{100} i$, which is $\frac{1}{50} \cdot 3775$. So the expected number of inch markers on the long end of the stick is 75.5