Homework 11

Solution Key

All homeworks are due at 1:00pm in the CS22 bin on the CIT second floor, opposite the elevators.

Write your full name *and the problem number on each piece of paper you hand in and then staple.*

Reading*: Class notes chapters 1, 2, 3, 4.*

Note: *Please show your work and explain your reasoning on each problem in this assignment. We cannot award partial credit to students for incorrect answers that do not show their work, nor can we give full credit to students for correct answers that do not show their work.*

Problem 11.1

A space probe communicated with earth using bit strings, which contain a 1 $\frac{2}{3}$ *of the time and a* 0 $\frac{1}{3}$ *of the time. A transmitted* 1 *received as* 1 *with probability 0.9 (and, of course, as* 0 *with probability* 0.1*). A transmitted* 0 *is received as a 0 with probability* 0.8*. Given that a* 0 *is received, what is the probability that a* 0 *was transmitted?*

Problem 11.2

Suppose the number of visitors to a museum each day has Poisson distribution with parameter θ. Further assume that θ varies from day to day according to a uniform distribution on $[100, 300]$*. What is the expected number of visitors on a randomly selected day? (Hint: Use $E[E[X|Y]] = E[X]$)*

Let us call the number of visitors in a day Y, and the parameter of the Poisson distribution Θ. Since Y is a Poisson random variable, we have

$$\Pr(Y = y \mid \Theta = \theta) = \frac{\theta^y}{y!} * e^{-\theta}, (\theta > 0), \text{ mean: } \theta, \text{ variance : } \theta$$

From the above, we know that

$$E(Y \mid \Theta = \theta) = \theta$$

Recall that

$$E(Y) = E[E(Y \mid \Theta = \theta)] = E(\Theta)$$

Because Θ is uniformly distributed on $[100, 300]$, we then know that

$$E(Y) = E(\Theta) = \frac{100 + 300}{2} = 200$$

Problem 11.3

Two players A and B play the following game. A coin with probability p of turning up heads is tossed repeatedly. Player A wins the game if heads appears at least m times before tails has appeared n times; otherwise, player B wins the game. Find the probability that player A wins the game.

The minimum number of trials needed to win, then, is m. The maximum until a winner is determined is $m+n-1$. Let t be the number of trials until a win. For the game to end, Plater A must flip the last head, thus ending the game. This event occurs with a probability p. In the $t-1$ trials we must distribute between $i = 0$ to $i = n-1$ tails. Thus the probability of

$$\Pr[A \text{ wins}] = \sum_{t=0}^{n-1} \binom{m+t}{t} p^{m-1}(1-p)^t * p$$

Or more simply:

$$\Pr[A \text{ wins}] = \sum_{t=0}^{n-1} \binom{m+t}{t} p^{m}(1-p)^t$$

Alternatively, if we index our probability using the number of flips, we can vary between m flips to $m+n-1$ flips. If we consider i the number of events in the trial, the sum of the events in which a wins is then:

$$\Pr[A \text{ wins}] = \sum_{i=m}^{m+n-1} \binom{i}{m} p^{m}(1-p)^{i-m}$$

Problem 11.4

Suppose 1 out of 10,000 tickets is a winning ticket in a state lottery. How many tickets does one need to purchase so that he/she has at least 50% chance to have a winning ticket? (Use Poisson approximation)

P(chance of having at least one winning ticket) $\geq 50\%$
$n =$ number of tickets to buy
$x =$ number of winning tickets
X is $B(n, \frac{1}{10000})$

n should be large enough that $P(x = 1) \geq 50\%$.

$$
\begin{aligned}
P(x=1) \geq 50\% \quad &\iff \quad P(x=0) \leq 50\% \\
&\iff \quad e^{-\lambda} * \frac{\lambda^0}{0!} = e^{-\lambda} \leq 50\% \\
&\iff \quad -\lambda \leq \ln(0.5) \\
&\iff \quad \lambda \geq \ln(2) \\
&\iff \quad \lambda = \frac{n}{10000} \geq \ln(2) = 0.69
\end{aligned}
$$

Therefore $n = 6900$.

Problem 11.5

Suppose John and Mary each tosses a fair coin twice. Let X be the number of heads John obtains and Y the number of heads Mary obtains.

a. What is the distribution of X? What is the distribution of Y?

b. What is the distribution of $X + Y$?

c. Given $X + Y = 2$, find the distribution and expected value of X.

a.

$$P(X = k) = \binom{2}{k}(p^k)(1-p)^{2-k} = \binom{2}{k}\left(\frac{1}{2}\right)^2, \ 0 \leq k \leq 2$$

$$P(Y = k) = \binom{2}{m}(1-p)^{2-m} = \binom{2}{m}\left(\frac{1}{2}\right)^2, \ 0 \leq m \leq 2$$

$\therefore$ X and Y are both binomial distributions with $p = \frac{1}{2}$.

b. $X + Y$: Same as tossing a fair coin 4 consecutive times

$$P(X+Y=k) = \binom{4}{k}(p^k)(1-p)^{4-k} = \binom{4}{k}\left(\frac{1}{2}\right)^4$$

$\therefore X + Y$ is a binomial distribution with $p = \frac{1}{2}$.

c.

$$\begin{aligned}
P(X=0 \mid X+Y=2) &= \frac{P(x=0)*P(y=2)}{P(x+y=2)} \\
&= \frac{(\binom{2}{0}*p^0*(1-p)^2)*(\binom{2}{2}*p^2*(1-p)^2)}{\binom{4}{2}*p*(1-p)^2} \\
&= \frac{(1-p)^2*(1-p)^2}{6*p^2*(1-p)^2} \\
&= \frac{(1-p)^2}{6*p^2} \\
&= \frac{1}{6} \\
P(X=1 \mid X+Y=2) &= \frac{P(x=1)*P(y=1)}{P(x+y=2)} \\
&= \frac{2*p*(1-p)*2*p*(1-p)}{6*p^2*(1-p)^2} \\
&= \frac{2}{3} \\
P(X=2 \mid X+Y=2) &= \frac{p(x=2)*P(y=0)}{P(x+y=2)} \\
&= \frac{(1-p)^2*(1-p)^2}{6*p^2*(1-p)^2} \\
&= \frac{1}{6} \\
E(X) &= \frac{1}{6}*0+\frac{2}{3}*1+\frac{1}{6}*2 = 1
\end{aligned}$$

Problem 11.6

Suppose that the number of tin cans recycled in a day at a recycling center is a random variable with an expected value of $50,000$ *and a variance of* $2,500$.

a. *Find an upper bound on the probability that the center will recycle more than* 55,000 *cans on a particular day.*

b. *Find a lower bound on the probability that the center will recycle between* 38,000 *and* 62,000 *cans on a particular day.*