CSCI 0220 **Discrete Structures and Probability** B. Raphael

Practice Final Exam

3 May 2008

Problem 1

Express the following statements as predicates, quantified as necessary. The variable $n \in \mathbf{N}$ should not be quantified.

- n is an integral multiple of 2 and of 7.
- n is not the square of an integer.
- For all integers > 2, there is exists a prime that divides j (use Prime(x): "x is prime").

Problem 2

Prove that for all $n \geq 0$, $2^{n+2} + 3^{2n+1}$ is a multiple of 7.

Problem 3

Let $G = (V, E)$ with $\mid V \mid = n$ be an undirected graph.

Prove the equivalence of the following statement (i.e. one implies that other and conversely):

G is acyclic and $\mid E \mid = n - 1 \Longleftrightarrow G$ is connected but becomes disconnected if an edge is removed from E.

Problem 4

Let t_n be the number of strings of length n on $\{0, 1\}$ that do not contain 11 as a substring.

- Develop a recurrence relation for t_n.
- Solve the recurrence relation (satisfying the initial conditions).

Problem 5

Franco has noticed that there have been some errors in the homework problems. There are errors in 0.1% of all problems. He's charged two TAs with finding the flawed problems.

a. Leo has eagle-eyes when it comes to finding errors in problems. If there's a mistake, he'll detect it 98% of the time. However in his zeal he'll claim a problem contains a mistake 4% of the time, even if it's correct.

   What is the probability that Leo correctly identifies a flawed problem?

b. Brendan, on the other hand, finds work distasteful. He simply states the every thousandth problem contains a flaw, even if it doesn't. What's the probability that Brendan correctly identifies a flawed problem?

c. Which TA correctly identifies flawed problems with greater probability?

Problem 6

On average, 15 out-of-state cars pass a certain point on the highway per hour. What is the probability that exactly four out-of-state cars pass that point in a 12-minute period?

Problem 7

As you may know, your friendly CS22 TAs get together every week to eat pizza (and grade your homework). Last week, Antonio's had a promotional sale that the TAs could not pass up. Their commercial reads:

"Buy 2 large pizzas at the regular price, and for each pizza get up to 11 different toppings (from the 11 possible) absolutely free. (But you cannot have the same topping appear more than once on the same pizza.) That's 4,194,304 different ways to design your order!"

The writer of the ad was no stranger to counting. He reasoned that the number of ways to choose different toppings for one pizza is all the possible subsets of the set of 11 toppings, and there are 2^{11} such subsets. Since there are two pizzas, he reasoned that the total possible combinations of pizzas is $(2^{11})^2 = 4,194,304$.

Unfortunately, the writer of the ad suffered the grave misfortune of never having taken CS22, and alas, the number $(2^{11})^2$ is actually wrong. Can you help him get it right?

a. Explain what is wrong with the ad writer's counting.

b. In how many ways can you really choose toppings for the two pizzas (a formula is good enough, no need to calculate the exact number)?

c. In how many ways could you choose toppings for n pizzas? (Hint: use stars and bars.)

d. Suppose that 3 of the 11 possible toppings are meat toppings. Suppose that we order *one* pizza and choose its set of toppings uniformly at random out of the total set of possibilities. What is the probability that Josh, our lone vegetarian TA, will be able to eat it? (Luckily, Josh would eat any pizza as long as there's no meat on it.)

Problem 8

a. Recall the Binomial Theorem: Let x and y be variables, and let n be a nonnegative integer. Then

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i$$

What is the coefficient of $x^{20}y^5$ in the expansion of $(x+y)^{25}$? You may leave choose notation in your response.

b. You are drawing two cards from a deck of playing cards. Find the probability that the second card you draw will be a spade, given that the first one was a spade.

c. Let $A = \{a, b, c, d\}$
Let $B = \{2, 3, 4, 5, 6\}$
Let $f : A \to B$ be the function defined as follows:

$$\{(a,2), (b,3), (c,5), (d,6)\}$$

So, for example, $f(b) = 3$. Is f surjective? Justify your answer with a sentence or two in terms of the set definition of f.

d. Again, use the definitions given in part *c*. Is f surjective? This time, justify your answer with a sentence or two in terms of the cardinalities of A and B.

e. Let the random variable X represent the number of times you roll a fair, 6-sided die before seeing each face at least once. Assume that $E[X] = 14.7$. Using this result and Markov's inequality, how many times do you have to roll the die in order to be 99% sure you've seen all 6 sides?

Problem 9

One day, you try to view the immensely popular website `www.cs.brown.edu/courses/cs022/`. Unfortunately, the site is so popular that each time you attempt to connect to it, you have only a $p \leq 1$ probability of achieving a connection; the rest of the time, you receive an error message. Each attempt at a connection is independent of every other attempt. Undeterred, you decide to reload the page over and over until you succeed in viewing the site.

a. Let the random variable X represent the number of times you attempt to connect before you succeed. Find $E[X]$. Prove your answer using the formula for an infinite geometric series: $\sum_{i=0}^{\infty} x^i = \frac{1}{1-x}$.

b. What is the probability that your first attempt is successful?

   What does the number $1 - p$ represent?

   What is the probability that your second attempt is successful, given that your first one wasn't?

   Explain, using your understanding of probability and expected value, why the equation

$$E[X] = p + (1 - p)(1 + E[X])$$

   provides an alternate way to solve part a.

c. Bound $\Pr[X \geq 2/p]$, the probability that you will receive an error message at least $2/p$ times before you succeed in viewing the site. Explain what formulae you used, and why.

Problem 10

How many bit strings of length 10 contain either five consecutive `0`'s or five consecutive `1`'s?

Problem 11

Prove that in a group of 25 people, each person cannot shake hands with exactly three other people. (Hint: use a graph.)

Problem 12

Let S be a set of ten integers chosen from 1 to 50. Use the pigeonhole principle to show that the set contains at least two different (but not necessarily disjoint) subsets of four integers that add up to the same number. (For example, if $S = \{3, 8, 9, 18, 24, 34, 35, 41, 44, 50\}$ then the subsets can be $\{8, 24, 34, 35\}$ and $\{9, 18, 24, 50\}$ which both sum to 101.) Hint: Consider the function $F : \mathcal{P}(S) \rightarrow T$ where T is the set of all possible sums of elements of S. What are the cardinalities of $\mathcal{P}(S)$ and T?