CSCI 0220 **Discrete Structures and Probability** F. Preparata

Recitation 3

Solution Key

Problem 3.1

Consider the following recurrence relation:

$$a_{n+1} = a_n + 2a_{n-1}$$

Given the initial conditions $a_0 = 1$ *and* $a_1 = 2$*, solve the recurrence relation by*

a. *iteration (prove that your guess is correct)*

b. *the characteristic equation*

Now consider $a_{n+1} = a_n + 2a_{n-1} + 10$*. This is known as an inhomogeneous recurrence relation, since there is a term not involving elements of the sequence. However, it can be rewritten as a linear homogeneous recurrence relation (what we have covered in class)!*

c. *Consider* $a_{n+2} - a_{n+1}$*. Write a linear recurrence relation based on the simplification of that expression.*

d. *Write the characteristic equation and find the roots. How do the roots relate to the homogeneous (original) case?*

a. The sequence goes $\{1, 2, 4, 8, 16, \dots\}$. This is $a_n = 2^n$. Proof:

$$2^n + 2 \cdot 2^{n-1} = 2 \cdot 2^n = 2^{n+1}$$

b. $r^2 - r - 2 = (r+1)(r-2)$. This implies that $a_n = c_1(-1)^n + c_2 2^n$. From the initial conditions, we find that $c_1 = 0$ and $c_2 = 1$.

c.

$$\begin{aligned} a_{n+2} - a_{n+1} &= a_{n+1} + 2a_n + 10 - a_n - 2a_{n-1} - 10 \\ &= a_{n+1} + a_n - 2a_{n-1} \end{aligned}$$

Thus:

$$a_{n+2} = 2a_{n+1} + a_n - 2a_{n-1}$$

The characteristic equation is $r^3 - 2r^2 - r + 2 = 0$. Since the coefficients sum to 0, we know that 1 must be a root. Thus, $r^3 - 2r^2 - r + 1 = (r-1)(r^2 - r - 2)$. The quadratic polynomial is exactly the same one from before (thus its roots remain the same). This implies that $a_n = c_1 1^n + c_2 2^n + c_3(-1)^n$. It's the same as the homogenous solution, plus a constant. (The original equation also happens to be the homogeneous relation plus a constant – this is no coincidence. Look up the method of undetermined coefficients for more on this.)

Problem 3.2

We have a rack of the 15 balls used in a typical game of billiards. Each ball has a number from 1-15, the first 8 balls are solid colors, the second 7 balls are striped.

a. *How many ways can you select 4 solid-colored balls and 4 striped balls? (Remember that each ball has a unique number...)*

b. *How many ways can you select a group of (at least one) prime-numbered balls?*

a. First we select 4 solid-colored balls from the 8, which can be done in $\binom{8}{4}$ ways. Then we select 4 striped balls from the 7 available, which can be done in $\binom{7}{4}$. Therefore the total number of ways of choosing all 8 balls is $\binom{8}{4}\binom{7}{4} = 2450$.

b. First we note that the prime numbered balls are: 2,3,5,7,11,13. So, to enumerate all possible groups we must add up selections of size 1,2,3,4,5,6.
For each group size, n, there are $\binom{6}{n}$ ways to create groups of the given size from the 6 prime-numbered balls. By adding up all the group sizes we get:
$\binom{6}{1} + \binom{6}{2} + \binom{6}{3} + \binom{6}{4} + \binom{6}{5} + \binom{6}{6}$
$= 6 + 15 + 20 + 15 + 6 + 1 = 63$ ways.

Problem 3.3

Among 20 students in a room, 7 study applied mathematics, 10 study economics, and 10 study computer science. Also, 3 study applied mathematics and economics, 4 study applied mathematics and computer science, and 6 study economics and computer science. We know 2 students study all three subjects. How many of the 20 students in the room study none of the three subjects?

Using the inclusion-exclusion principle, and letting
A = [students studying applied mathematics]
B = [students studying economics]
C = [students studying computer science]
$\{A \cup B \cup C\} = |A| + |B| + |C| - |A \wedge B| - |A \wedge C| - |B \wedge C| + |A \wedge B \wedge C|$
$\{A \cup B \cup C\} = 7 + 10 + 10 - 3 - 4 - 6 + 2 = 16$
Since there are 20 students total, and 16 of them study at least one of applied mathematics, economics, and computer science, we deduce that 4 students must study none of the three subjects.

Problem 3.4

Carmen Sandiego is sending an encrypted message back describing each attribute of Wonder Rat, a treacherous villain. Since each attribute has three choices, he represents Wonder Rat's profile as a ternary string. A ternary string is a finite string composed over the alphabet $\{a, b, c\}$. How many ternary strings of length 9 have at least two `a`'s, one `b`, and four `c`'s? Please calculate the numerical result.

Consider a string with two zeroes, one one and four twos. Such a string satisfies the required elements of the string given in the problem, so the remaining three characters in the string can be any number from the set $\{a, b, c\}$.

$$\begin{array}{ccccccccc} a & a & b & c & c & c & c & & \\ - & - & - & - & - & - & - & - & - \end{array}$$

The remaining two slots can be filled as follows:

a	b	c	Permutations	
2	1	6	$\frac{9!}{2!1!6!}$	252
3	1	5	$\frac{9!}{3!1!5!}$	504
4	1	4	$\frac{9!}{4!1!4!}$	630
2	2	5	$\frac{9!}{2!2!5!}$	756
3	2	4	$\frac{9!}{3!2!4!}$	1260
2	3	4	$\frac{9!}{2!3!4!}$	1260

Totalling all of these terms gives us the answer: 4662.