CSCI 0220 **Discrete Structures and Probability** F. Preparata

Recitation 3

Solution Key

Problem 3.1

CS022 has been known to cause hypertension for the students taking the course. It is found that a particular drug is successful in reducing the blood pressure of students suffering from hypertension in 60 *percent of cases. Five students are suffering from hypertension in health services are to be treated with the drug. What is the probability that more than three students have their blood pressure reduced?*

Problem 3.2

The physics department at Brown has recently been trying to create a new element known as Bruonium, by colliding particles together in a particle accelerator that lies underneath Keeney. In 8 *percent of the collisions, a certain type of fundamental particle can be detected. If 60 collisions are observed, use the Poisson approximation to find the probability that the fundamental particle is detected in exactly* 5 *experiments.*

Problem 3.3

You and your friend are in a heated debate about which of two restaurants to go to for dinner. To settle the argument, you propose to flip a coin. Your friend objects on the grounds that the Canadian quarter in your pocket is biased. To prove her point, she tells you to flip the coin until 50 heads appear. If the number of tails that has appeared at this point is not 50, then she claims that the coin must be biased.

a. Let X be the number of flips until n heads appear. Find $Pr[X = i]$.

b. What is the probability that your friend will agree that the coin is not biased?

a. We are looking for precisely m heads in X trials. So, there must be $m-1$ heads in the first $x-1$ trials, followed by a head for the xth trial. Thus:

$$Pr(X = x) = \left[\binom{x-1}{m-1} p^{m-1}(1-p)^{(x-1)-(m-1)}\right] p = \binom{x-1}{m-1} p^m (1-p)^{x-m}$$

This distribution is called the *negative binomial distribution.*

b. Exactly 50 heads and 50 tails implies that the number of trials is 100. Thus:

$$Pr(X = 100) = \binom{99}{49} (0.5)^{50}(0.5)^{50} \approx 0.0398$$

Problem 3.4

On average, 15 out-of-state cars pass a certain point on the highway per hour. What is the probability that exactly four out-of-state cars pass that point in a 12-minute period?

Since 15 cars pass in an hour on average, $\frac{15}{5} = 3$ cars pass in a 12 minute interval on average. The number X of cars that pass in a 12 minute period is a Possion random variable with mean $\lambda = 3$. Thus,

$$Pr[X = 4] = \frac{3^4}{4!} e^{-3} \approx 0.168.$$

Problem 3.5

Suppose that an airplane engine fails in flight with probability $1 - p$, independently for each engine of the plane. An airplane can still fly if $\geq 50\%$ of the engines are operative. For what values of p is a 4-engine plane preferable to a 2-engine plane?

The total number of operational engines can be viewed as a binomial distribution, with n being the number of engines, and p being the probability of remaining in operation.

A 2-engine plane crashes if and only if both engines fail (0 successes and 2 failures). This has probability $\binom{2}{0} p^0 (1-p)^2 = (1-p)^2$.

A 4-engine plane crashes if and only if 3 or 4 engines fail. (1 success and 3 failures, or 0 success and 4 failures, respectively). 2 failures still keeps the plane in flight. This has probability $\binom{4}{0} p^0 (1-p)^4 + \binom{4}{1} p^1 (1-p)^3 = (1-p)^4 + 4p(1-p)^3$.

We want to know when the 4-engine plane is preferable; this means that the probability of crashing is less for a 4-engine plane than a 2-engine plane. This is equivalent to solving the following inequality:

$$(1-p)^2 > (1-p)^4 + 4p(1-p)^3$$

Let $q = 1-p$. Then we have $q^2 > q^4 + 4(1-q)q^3 \implies 0 > -3q^4 + 4q^3 - q^2$. Let us consider the polynomial $f(q) = -3q^4 + 4q^3 - q^2$. Its coefficients sum to 0, so 1 must be a root. We also notice that q^2 is present in each term; therefore 0 is also a root. This allows us to factor the polynomial to $f(q) = q^2(q-1)(-3q+1)$. This has the only relevant solution at $q = \frac{1}{3}$. Using sign analysis, we see that $f(q)$ is negative for $0 < q < \frac{1}{3}$ (consider $f(0.1) = -0.0003 + 0.0040 - 0.0100 = -0.0063$). Since $q = 1-p$, we have $\frac{2}{3} < p < 1$.