

## Recitation 5

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### Problem 5.1

Suppose  $f : \mathbb{R} \rightarrow \mathbb{R}$  and let  $c \in \mathbb{R}$ . Consider the following statement.

If for all  $\epsilon > 0$  there exists  $\delta > 0$  such that  $|x - c| < \delta \implies |f(x) - f(c)| < \epsilon$ , then  $f$  is continuous at  $c$ .

[Recall that  $p \implies q$  is equivalent to “if  $p$  then  $q$ ”.]

- (a) Write (in English) the converse of this statement.
- (b) Write (in English) the contrapositive of this statement. (Simplify your answer so that it does not include a statement of the form  $\sim (p \implies q)$ ).

### Problem 5.2

Prove that 3 divides  $n^3 + 2n$  whenever  $n > 0$  using induction.

Write a second proof which uses case analysis instead.

### Problem 5.3

Prove that a tree containing  $n$  nodes must have exactly  $n-1$  edges.

### Problem 5.4

A circular disk is cut into  $n$  distinct sectors, each shaped like the piece of a pie and all meeting at the center point of the disk. Each sector is to be painted red, green, yellow, or blue in such a way that no two adjacent sectors are painted the same color. Let  $S_n$  be the number of ways to paint the disk.

- (a) Derive the recurrence relation  $S_k = 2S_{k-1} + 3S_{k-2}$  for  $k \geq 4$ .

Hint: For 3 adjacent sectors of the disk, consider two possible cases

1. Sectors 1 and 3 are painted the same color.
  2. Sectors 1 and 3 are painted different colors.
- (b) Solve the recurrence in part (a). Note: You must determine the initial conditions  $S_2$  and  $S_3$ .

**Problem 5.5**

Franco has noticed that there have been some errors in the homework problems. There are errors in 0.1% of all problems. He's charged two TAs with finding the flawed problems.

- a. Ben has eagle-eyes when it comes to finding errors in problems. If there's a mistake, he'll detect it 98% of the time. However in his zeal he'll claim a correct problem contains a mistake 4% of the time.  
What is the probability that Ben correctly identifies a flawed problem?
- b. Alex, on the other hand, finds work distasteful. He simply states the every thousandth problem contains a flaw, even if it doesn't. What's the probability that Alex correctly identifies a flawed problem?
- c. Which TA correctly identifies flawed problems with greater probability?

**Problem 5.6**

For this problem, we assume  $X, Y, Z$  are sets, and  $f, g$  are functions such that  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$ .

- Prove that if  $g \circ f$  is injective, then  $f$  is injective.
- Prove that if  $g \circ f$  is surjective, then  $g$  is surjective.

**Problem 5.7**

Suppose that in a certain state, all automobile license plates have four letters followed by three digits.

- How many different license plates are possible?

- How many license plates could begin with A and end in 0?
- How many license plates are possible in which all the letters and digits are distinct?

**Problem 5.8**

On average, 15 out-of-state cars pass a certain point on the highway per hour. What is the probability that exactly four out-of-state cars pass that point in a 12-minute period?

**Problem 5.9**

Suppose that a random simple graph  $G$  is constructed from a set of  $n$  vertices by joining each pair of nonidentical vertices by an edge with probability  $p$ . For each vertex  $v$  let  $X_v$  equal the degree of  $v$ , so  $X_v$  is a random variable.

- What is  $Pr[X_v = k]$  for  $k \geq 0$ ?
- What is  $E[X_v]$ ?
- Suppose  $G$  has one million vertices and  $p = \frac{1}{2}$ . Use Chebyshev's inequality to give an upper bound on the probability that a vertex  $v$  has degree  $\geq 501000$  or  $\leq 499000$ ?

**Problem 5.10**

In a 12-day period, a small business mailed 195 bills to customers. Show that during some period of three consecutive days at least 49 bills were mailed.