CSCI 0220 Discrete Structures and Probability F. Preparata

Recitation 5

Solution Key

Problem 5.1

Suppose $f : \mathbb{R} \to \mathbb{R}$ *and let* $c \in \mathbb{R}$. *Consider the following statement.*

If for all $\epsilon > 0$ *there exists* $\delta > 0$ *such that* $|x - c| < \delta \implies |f(x) - f(c)| < \epsilon$, *then* f *is continuous at* c.

[Recall that $p \implies q$ *is equivalent to "if* p *then* q*".]*

(a) Write (in English) the converse of this statement.

(b) Write (in English) the contrapositive of this statement. (Simplify your answer so that it does not include a statement of the form $\sim (p \implies q)$*).*

(a) "If f is continuous at c, then for all $\epsilon > 0$ there exists $\delta > 0$ such that $|x - c| < \delta$ implies $|f(x) - f(c)| < \epsilon$."

(b) "If f is discontinuous at c, then there exists $\epsilon > 0$ such that for all $\delta > 0$, $|x - c| \leq \delta$ and $|f(x) - f(c)| \geq \epsilon$"

Problem 5.2

Prove that 3 divides $n^3 + 2n$ *whenever* $n > 0$ *using induction.*

Write a second proof which uses case analysis instead.

- First, we try the induction approach.

  Basecase: $n = 0$, $n^3 + 2n = 0$, which is divisible by 3. Now, we assume that $p(x)$ is true, i.e. $x^3 + 2x = 3k$ where k is an integer. We want to prove that $(x+1)^3 + 2(x+1) = 3m$ where m is also an integer.

  We can factor out the equation to be the following $x^3 + 3x^2 + 3x + 1 + 2x + 2$. Now, substitute $3k$ for $x^3 + 2x$, we get $3k + 3x^2 + 3x + 3$. As we can see there is a factor of 3 in every term of the addition, so the sum is divisible by 3.

- Now, we try to prove by case analysis. This approach is not as intuitive as some of the case analysis examples you have seen. Usually, you want to assume that the sum of the entire equation is equal to a set of different cases, and then go about proving the truthfulness of those cases. However, in this case, you want to identify the different categories for which n could belong to.

  - $n = 3k$ where k is an integer.
    Now, substituting $3k$ for n, we get $3k^3 + 2 * 3k$, which is divisible by 3.
  - $n = 3k + 1$ where k is an integer.
    Substituting again, we get $27k^3 + 27k^2 + 9k + 3k + 3$. Each term in the sum has a coefficient that is divisible by 3, so the sum is as well.
  - $n = 3k + 2$ where k is an integer.
    Substituting again, we get $27k^3 + 54k^2 + 36k + 3k + 12$. Again, each term in the sum has a coefficient that is divisible by 3, so the sum is as well.

  Since all integers greater than 0 can be placed into one of these three categories, we have proved that this property holds true for every non-negative integer.

Problem 5.3

Prove that a tree containg n nodes must have exactly n-1 edges.

We will use induction to solve this one. Consider the base case where there is a tree with only one node. That one node will be the root, and no edges will be coming out of it, so the tree contains 1 node, and $0 = 1 - 1$ edges. The definition of a tree is essentially an acyclic connected graph. This means there is only one path between any two vertices. Removal of any edge will disconnect it. This is true because suppose we have a tree G with an edge, e = (u,v), removed between two vertices, u and v. In order for the graph to remain a tree, it must remain connected, this means a path exists between all vertices, so there must remain a path from u to v. Replacing that single edge e obtains two distinct paths from u to v, which forms a cycle, which contradicts the definition of a tree. So removal of an edge in a tree disconnects the tree into two subgraphs between the nodes where the edge was removed. Let these two subgraphs be G_u, G_v, where G_u contains all nodes previously in G with paths to u alone and G_v contains all nodes previously in G with paths to v alone. Since there existed no cycles, the subgraphs are trees. Let each subgraph G_u and G_v have n_u and $n_v nodes$, respectively. According to the inductive hypothesis, each subgraph G_u, and G_v have $n_u - 1$, and $n_v - 1$ edges. Then adding the edge between u and v, creates a new connected tree with $n = n_u + n_v$ nodes, and $n - 1 = n_u - 1 + n_v - 1 + 1$ edges. Our inductive hypothesis has been extended, and a tree contains n nodes with n-1 exactly edges.

Problem 5.4

A circular disk is cut into n distinct sectors, each shaped like the piece of a pie and all meeting at the center point of the disk. Each sector is to be painted red, green, yellow, or blue in such a way that no two adjacent sectors are painted the same color. Let S_n be the number of ways to paint the disk.

(a) *Derive the recurrence relation $S_k = 2S_{k-1} + 3S_{k-2}$ for $k \geq 4$.*

Hint: For 3 adjacent sectors of the disk, consider two possible cases

1. *Sectors 1 and 3 are painted the same color.*
2. *Sectors 1 and 3 are painted different colors.*

(b) *Solve the recurrence in part (a). Note: You must determine the initial conditions S_2 and S_3.*

(a) If sectors 1 and 3 are painted the same color, then consider a disk with sectors 1-3 combined into one. There are S_{k-2} ways to color this disk. For each of these colorings ther are 3 ways to color sector 2, since 2 must be colored differently from 1 and 3. Thus, there are $3S_{k-2}$ colorings in this case.

If sectors 1 and 3 are painted different colors, we can remove sector 2 and paint the remaining $k - 1$ sectors in S_{k-1} ways. For each of these colorings, we can color sector 2 in one of the two colors not chosen for sectors 1 and 3. Thus, there are $2S_{k-1}$ colorings in this case.

Cases 1 and 2 are disjoint, so the total number of colorings is $S_k = 2S_{k-1} + 3S_{k-2}$.

(b) To solve this recurrence relation, we must determine the characteristic equation and initial conditions. The characteristic equation is

$$r^2 - 2r - 3 = 0$$

Solving for r (by factoring or the quadratic formula), we get that $r = 3$ or $r = 1$.

The initial conditions for this recurrence are S_2 and S_3. A disk with 2 sectors can be colored using any two of the 4 colors in any order, or $S_2 = \binom{4}{2} * 2! = 12$. A disk with 3 sectors can be colored using any three of the 4 colors in any order, or $S_3 = \binom{4}{3} * 3! = 24$. We must now solve for the coefficients c_1 and c_2 in the equation below, using the initial conditions.

$$S_k = c_1 * 3^k + c_2 * 1^k = c_1 3^k + c_2$$

$$S_2 = 9c_1 + c_2 = 12$$

$$S_3 = 27c_1 + c_2 = 24$$

$$18c_1 = 12 \Rightarrow c_1 = \frac{2}{3} \Rightarrow 9\left(\frac{2}{3}\right) + c_2 = 12 \Rightarrow c_2 = 6$$

We now have the explicit formula:

$$S_k = 9 * 3^k + 6$$

Problem 5.5

Franco has noticed that there have been some errors in the homework problems. There are errors in 0.1% of all problems. He's charged two TAs with finding the flawed problems.

a. *Ben has eagle-eyes when it comes to finding errors in problems. If there's a mistake, he'll detect it* 98% *of the time. However in his zeal he'll claim a correct problem contains a mistake* 4% *of the time.*

   What is the probability that Ben correctly identifies a flawed problem?

b. *Alex, on the other hand, finds work distasteful. He simply states the every thousandth problem contains a flaw, even if it doesn't. What's the probability that Alex correctly identifies a flawed problem?*

c. *Which TA correctly identifies flawed problems with greater probability?*

Let M be the event that there is a mistake in the problem, L be the event that Ben says there is a mistake, and B be the event that Alex says there is a mistake. We are given that $Pr[M] = 0.001$ and $Pr[M^c] = 1 - Pr[M] = 0.999$.

a. We have $Pr[L|M] = 0.98$ and $Pr[L|M^c] = 0.04$. By Bayes Theorem

$$\begin{aligned} P[M|L] &= \frac{Pr[L|M]Pr[M]}{Pr[L|M]Pr[M] + Pr[L|M^c]Pr[M^c]} \\ &= \frac{(0.98)(0.001)}{(0.98)(0.001) + (0.04)(0.999)} = 0.024. \end{aligned}$$

b. We have $Pr[B|M] = Pr[B|M^c] = 0.001$. Moreover $Pr[B] = 0.001$. Thus,

$$Pr[M|B] = \frac{Pr[M,B]}{Pr[B]} = \frac{Pr[B|M]Pr[M]}{Pr[B]} = \frac{(0.001)(0.001)}{0.001} = 0.001.$$

c. Thus, Ben is correct more often.

Problem 5.6

For this problem, we assume X, Y, Z are sets, and f, g are functions such that $f : X \to Y$ and $g : Y \to Z$.

- *Prove that if $g \circ f$ is injective, then f is injective.*
- *Prove that if $g \circ f$ is surjective, then g is surjective.*

- Assume $g \circ f$ is injective. Then for any $x_1, x_2 \in X$, if $g(f(x_1)) = g(f(x_2))$, then $x_1 = x_2$, by definition. We wish to show that if $f(x_1) = f(x_2)$, then $x_1 = x_2$. But if $f(x_1) = f(x_2)$, then applying g to both sides yields that $g(f(x_1)) = g(f(x_2))$, and therefore $x_1 = x_2$ by the injectivity of $g \circ f$. Thus, we have shown f is injective.
- Assume $g \circ f$ is surjective. Then by definition, for any $z \in Z$, there is some $x \in X$ such that $g(f(x)) = z$. Since $f : X \to Y$, we know that $f(x) \in Y$. Let $y = f(x)$. Then we know there is some $y \in Y$ such that $g(y) = z$. This is exactly what it means for g to be surjective.

Problem 5.7

Suppose that in a certain state, all automobile license plates have four letters followed by three digits.

- *How many different license plates are possible?*
- *How many license plates could begin with A and end in 0?*
- *How many license plates are possible in which all the letters and digits are distinct?*

Problem 5.8

On average, 15 *out-of-state cars pass a certain point on the highway per hour. What is the probability that exactly four out-of-state cars pass that point in a* 12*-minute period?*

Since 15 cars pass in an hour on average, $\frac{15}{5} = 3$ cars pass in a 12 minute interval on average. The number X of cars that pass in a 12 minute period is a Possion random variable with mean $\lambda = 3$. Thus,

$$Pr[X = 4] = \frac{3^4}{4!}e^{-3} \approx 0.168.$$

Problem 5.9

Suppose that a random simple graph G is constructed from a set of n vertices by joining each pair of nonidentical vertices by an edge with probability p. For each vertex v let X_v equal the degree of v, so X_v is a random variable.

a. *What is $Pr\left[X_v = k\right]$ for $k \geq 0$?*

b. *What is $E[X_v]$?*

c. *Suppose G has one million vertices and $p = \frac{1}{2}$. Use Chebyshev's inequality to give an upper bound on the probability that a vertex v has degree $\geq$ 501000 or $\leq$ 499000?*

a. We can think of an edge between a vertex v and the other vertices as a success and the absence of an edge as a failure, thus giving a binomial distribution. The probability of vertex v having k edges is p^k and the probability of v not being connected to the remaining $n-1-k$ vertices ($n-1$ since v cannot be connected to itself) is $(1-p)^{n-1-k}$. There are $\binom{n-1}{k}$ different ways of distributing the k edges among the $n-1$ vertices. Thus, we find:

$$Pr\left[X_v = k\right] = \binom{n-1}{k} p^k (1-p)^{n-k-1}$$

b. As we described in part (a) above, the random variable X_v follows the binomial distribution with parameters $n-1$ and p. Following the definition of the expected value of a binomially distributed random variable, we determine:

$$E[X_v] = (n-1)p$$

c. Using the fact that the degree of the vertices are binomially distributed, we calculate the mean degree of a vertex v as

$$\mu = (n-1)p = (10^6 - 1)\frac{1}{2} \approx 500000$$

Similarly, we can calculate the variance as

$$\sigma^2 = np(1-p) = 10^6 \cdot \frac{1}{2}(1 - \frac{1}{2}) \approx 250000$$

According to Chebyshev:

$$\Pr(|X - \mu| \geq \alpha) \leq \frac{\sigma^2}{\alpha^2}$$

Since we want to find the upper bound on the probability that a vertex v has degree $\geq$ 501000 or $\leq$ 499000, we use an α value of 1000. Substituting the values we calculated gives us:

$$\begin{aligned} \Pr(|X_v - 500000| \geq 1000) &\leq \frac{250000}{1000^2} \\ &\leq \frac{1}{4} \end{aligned}$$

Problem 5.10

In a 12-day period, a small business mailed 195 *bills to customers. Show that during some period of three consecutive days at least* 49 *bills were mailed.*

Let box 1 correspond to days $1, 2, 3$; let box 2 correspond to days $4, 5, 6$; box 3 to days $7, 8, 9$, and box 4 to days $10, 11, 12$. Putting each bill into the box corresponding to the day it was mailed, we see that one box must contain at least $\rceil \frac{195}{4} \lceil = 49$ bills, by the general form of the Pigeonhole principle.