Homework 5
Prop Logic

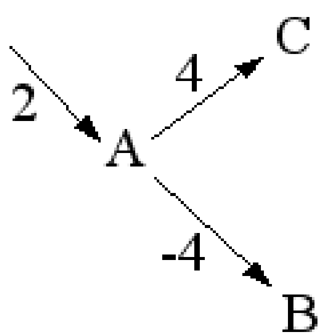
Due: 5:00pm on 3/14/08

Problem 5.1

Given the following SAT instance:

(-1 - 4 + 5) (-1 + 3 - 4) (+4 - 1 + 6) (+5 - 2) (+2 - 3 + 4) (-1 + 2) (-1 + 5 - 6) (+1 - 6) (+6 - 4 -5) (-1 - 6 + 4) (-3 - 2 - 4) (+4 + 1)

and the following branching graph:



For each node (A, B, and C) generate all clauses that you can learn and the associated inference graph. Use the results from node A when working on nodes B and C.

Solution:

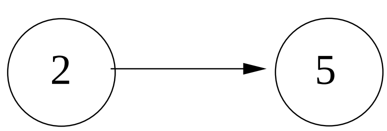
Already-satisfied clauses are marked with [SAT]. Assigned variables in unsatisfied clauses are marked with [X].

a. **Branch on** $2 = TRUE$

This gives us the following SAT instance:

(-1 -4 +5) (-1 +3 -4) (+4 -1 +6) (+5 [X-2]) [SAT(+2 -3 +4)] [SAT(-1 +2)] (-1 +5 -6) (+1 -6) (+6 -4 -5) (-1 -6 +4) (-3 [X-2] - 4) (+4 +1)

$2 = TRUE$ **forces us to Unit-Propogate** $5 = TRUE$**:**



Resulting SAT instance:

[SAT(-1 -4 +5)] (-1 +3 -4) (+4 -1 +6) [SAT(+5 [X-2])] [SAT(+2 -3 +4)]

[SAT(-1 +2)] [SAT(-1 +5 -6)] (+1 -6) (+6 -4 [X-5]) (-1 -6 +4)

(-3 [X-2] -4) (+4 +1)

And nothing further can be gained, so we have to branch again.

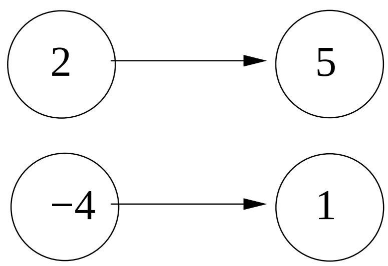
b. Branch on $4 = FALSE$

This gives us the following SAT instance:

[SAT(-1 -4 +5)] [SAT(-1 +3 -4)] ([X+4] -1 +6) [SAT(+5 [X-2])]

[SAT(+2 -3 +4)] [SAT(-1 +2)] [SAT(-1 +5 -6)] (+1 -6) [SAT(+6 -4 [X-5])] (-1 -6 [X+4]) [SAT(-3 [X-2] -4)] ([X+4] +1)

$4 = FALSE$ **forces us to Unit-Propogate** $1 = TRUE$**:**



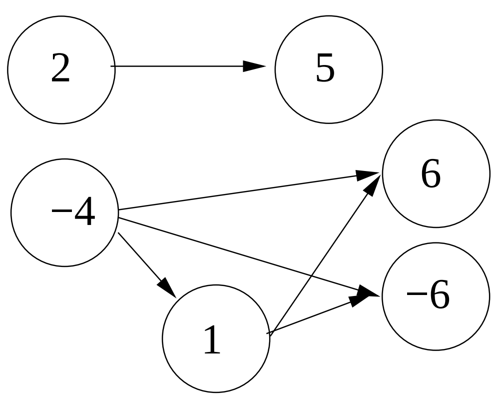
Resulting SAT instance:

[SAT(-1 -4 +5)] [SAT(-1 +3 -4)] ([X+4] [X-1] +6) [SAT(+5 [X-2])]

[SAT(+2 -3 +4)] [SAT(-1 +2)] [SAT(-1 +5 -6)] [SAT(+1 -6)]

[SAT(+6 -4 [X-5])] ([X-1] -6 [X+4]) [SAT(-3 [X-2] -4)] [SAT([X+4] +1)]

$1 = TRUE$ **and** $4 = FALSE$ **force us to propogate both** $6 = TRUE$ **and** $6 = FALSE$**:**



This is a contradiction. Backing up on our graph, the assumption that caused this was $4 = FALSE$**, so we can add the clause (4).**

c. "Branch" on $4 = TRUE$

This gives us the following SAT Instance:

[SAT(-1 -4 +5)] (-1 +3 [X-4]) [SAT(+4 -1 +6)] [SAT(+5 [X-2])]

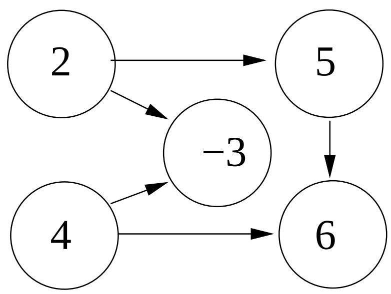
[SAT(+2 -3 +4)] [SAT(-1 +2)] [SAT(-1 +5 -6)] (+1 -6)

(+6 [X-4] [X-5]) [SAT(-1 -6 +4)] (-3 [X-2] [X-4]) [SAT(+4 +1)]

[SAT(+4)]

$4 = TRUE$ **and** $5 = TRUE$ **forces us to Unit-Propogate** $6 = TRUE$**, and**

$4 = TRUE$ **and** $2 = TRUE$ **forces us to Unit-Propogate** $3 = FALSE$**:**



Resulting SAT instance:

[SAT(-1 -4 +5)] (-1 [X+3] [X-4]) [SAT(+4 -1 +6)] [SAT(+5 [X-2])]

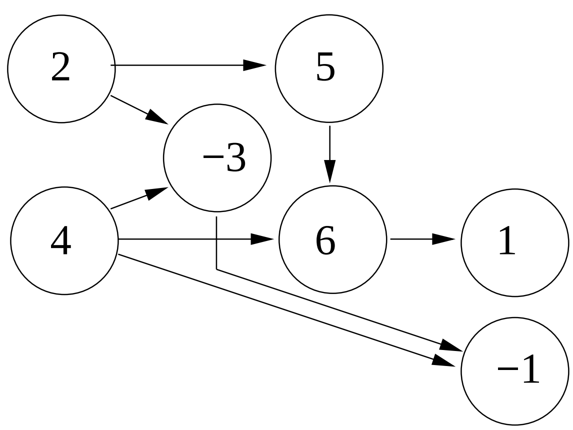
[SAT(+2 -3 +4)] [SAT(-1 +2)] [SAT(-1 +5 -6)] (+1 [X-6])

[SAT(+6 [X-4] [X-5])] [SAT(-1 -6 +4)] [SAT(-3 [X-2] [X-4])]

[SAT(+4 +1)] [SAT(+4)]

$6 = FALSE$ **forces us to Unit-Propogate** $1 = TRUE$**, and**

$3 = FALSE$ **and** $4 = TRUE$ **force us to Unit-Propogate** $1 = FALSE$



This is a contradiction. Backing up on our graph, the assumptions that caused this were $4 = TRUE$**, and** $2 = TRUE$**, so we can add the clause (-2 -4).**

Problem 5.2

Vertex coloring (aka map coloring) is the following problem: given a natural number k and a graph $G = (V, E)$ with vertices V and edges $E \subseteq V \times V$ such that $(v_1, v_2) \in E$ implies that $(v_2, v_1) \in E$, does there exist a way to assign one of k colors to each vertex such that no to adjacent vertices have the same color? That is, does there exist a function c from V to $\{n \in \mathbb{N} \mid 0 < n \leq k\}$, such that for all $(v_1, v_2) \in E$, $c(v_1) \neq c(v_2)$?

Give a polynomial-time reduction from vertex coloring to SAT. That is, show that given any instance of vertex coloring, that an instance of SAT can be constructed in polynomial time such that the clauses of that instance of SAT are satisfiable if and only if the vertex coloring is possible.

Solution:

For each $v \in V$ **and color** $c \in C = \{n \in \mathbb{N} \mid 0 < n \leq k\}$**, let** v_c **be a boolan variable which is true if** v **is colored** c **and false otherwise.**

The SAT instance consists of the conjunction of the following CNF form sentences:

$\bigwedge_{v \in V} \bigvee_{c \in C} v_c$ **(Each vertex must be at least one color) Time:** $O(|V|k)$

$\bigwedge_{v \in V} \bigwedge_{c,d \in C} \neg v_c \vee v_d$ **(If a vertex is one color, it is not any other color. In other words, each vertex can be at most one color.) Time:** $O(|V|k^2)$

$\bigwedge_{c \in C} \bigwedge_{v,u \in V} \neg v_c \vee \neg u_c$ **(Only one of two neighboring vertices can be the same color.) Time:** $O(k|V|^2)$

Problem 5.3

Given the knowledge base (KB) from last week's homework and a refutation complete algorithm for PL, how could you use the solver to show that the neutral element of any group over $\{0, 1\}$ is unique? How could you show that the inverse element is unique?

Solution:

Neutral Element unique:

$$KB \wedge \neg[X_{000} \wedge X_{101} \wedge X_{011} \wedge X_{010} \wedge X_{100} \wedge X_{111}]$$

Inverse unique:

$$(X_{000} \wedge X_{101} \wedge X_{011}) \wedge [X_{000} \wedge X_{000} \wedge (\neg[X_{010} \wedge X_{100}])] \wedge [X_{100} \wedge X_{010} \wedge (\neg[X_{110} \wedge X_{110}])] \vee$$

$$(X_{010} \wedge X_{100} \wedge X_{111}) \wedge [X_{001} \wedge X_{001} \wedge (\neg[X_{011} \wedge X_{101}])] \wedge [X_{101} \wedge X_{011} \wedge (\neg[X_{111} \wedge X_{111}])]$$