Homework 5
Logic and Learning

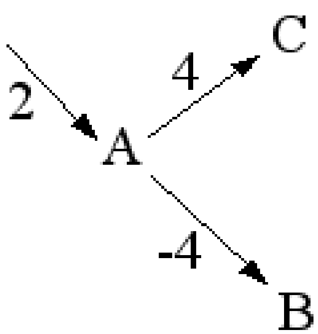
Due: 6:00pm on 10/28/09

Problem 5.1

Given the following SAT instance:

(-1 - 4 + 5) (-1 + 3 - 4) (+4 - 1 + 6) (+5 - 2) (+2 - 3 + 4) (-1 + 2) (-1 + 5 - 6) (+1 - 6) (+6 - 4 -5) (-1 - 6 + 4) (-3 - 2 - 4) (+4 + 1)

and the following branching graph:



For each node (A, B, and C) generate all clauses that you can learn and the associated inference graph. Use the results from node A when working on nodes B and C.

Solution:

Already-satisfied clauses are marked with [SAT]. Assigned variables in unsatisfied clauses are marked with [X].

a. Branch on $2 = TRUE$

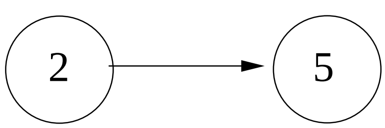
   This gives us the following SAT instance:

   (-1 -4 +5) (-1 +3 -4) (+4 -1 +6) (+5 [X-2]) [SAT(+2 -3 +4)]

   [SAT(-1 +2)] (-1 +5 -6) (+1 -6) (+6 -4 -5) (-1 -6 +4)

   (-3 [X-2] - 4) (+4 +1)

   $2 = TRUE$ forces us to Unit-Propogate $5 = TRUE$:

   Resulting SAT instance:

[SAT(-1 -4 +5)] (-1 +3 -4) (+4 -1 +6) [SAT(+5 [X-2])] [SAT(+2 -3 +4)]

[SAT(-1 +2)] [SAT(-1 +5 -6)] (+1 -6) (+6 -4 [X-5]) (-1 -6 +4)

(-3 [X-2] -4) (+4 +1)

And nothing further can be gained, so we have to branch again.

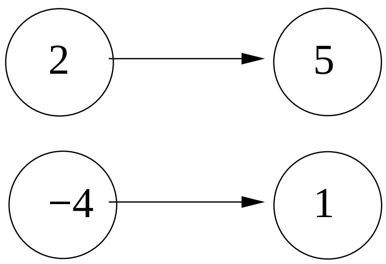
b. Branch on $4 = FALSE$

This gives us the following SAT instance:

[SAT(-1 -4 +5)] [SAT(-1 +3 -4)] ([X+4] -1 +6) [SAT(+5 [X-2])]

[SAT(+2 -3 +4)] [SAT(-1 +2)] [SAT(-1 +5 -6)] (+1 -6) [SAT(+6 -4 [X-5])] (-1 -6 [X+4]) [SAT(-3 [X-2] -4)] ([X+4] +1)

$4 = FALSE$ forces us to Unit-Propogate $1 = TRUE$:



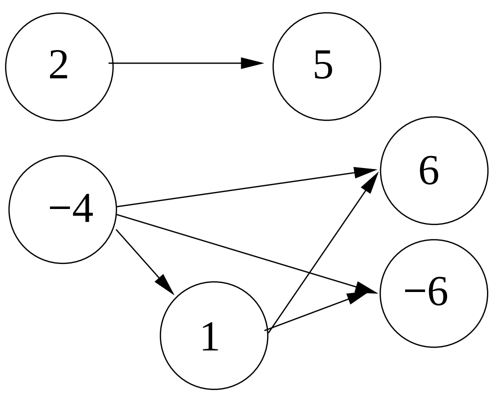
Resulting SAT instance:

[SAT(-1 -4 +5)] [SAT(-1 +3 -4)] ([X+4] [X-1] +6) [SAT(+5 [X-2])]

[SAT(+2 -3 +4)] [SAT(-1 +2)] [SAT(-1 +5 -6)] [SAT(+1 -6)]

[SAT(+6 -4 [X-5])] ([X-1] -6 [X+4]) [SAT(-3 [X-2] -4)] [SAT([X+4] +1)]

$1 = TRUE$ and $4 = FALSE$ force us to propogate both $6 = TRUE$ and $6 = FALSE$:



This is a contradiction. Backing up on our graph, the assumption that caused this was $4 = FALSE$, so we can add the clause (4).

c. "Branch" on $4 = TRUE$

This gives us the following SAT Instance:

[SAT(-1 -4 +5)] (-1 +3 [X-4]) [SAT(+4 -1 +6)] [SAT(+5 [X-2])]

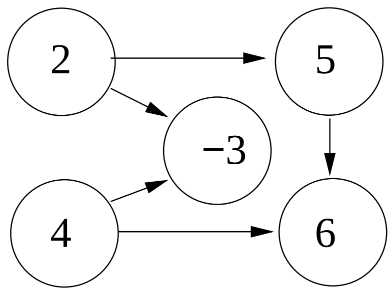
[SAT(+2 -3 +4)] [SAT(-1 +2)] [SAT(-1 +5 -6)] (+1 -6)

(+6 [X-4] [X-5]) [SAT(-1 -6 +4)] (-3 [X-2] [X-4]) [SAT(+4 +1)]

[SAT(+4)]

$4 = TRUE$ and $5 = TRUE$ forces us to Unit-Propogate $6 = TRUE$, and

$4 = TRUE$ and $2 = TRUE$ forces us to Unit-Propogate $3 = FALSE$:



Resulting SAT instance:

[SAT(-1 -4 +5)] (-1 [X+3] [X-4]) [SAT(+4 -1 +6)] [SAT(+5 [X-2])]

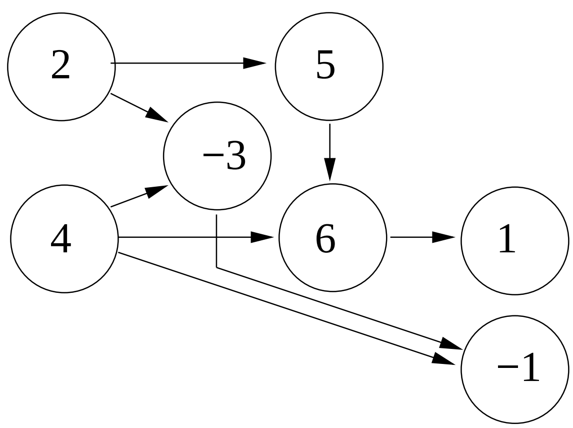
[SAT(+2 -3 +4)] [SAT(-1 +2)] [SAT(-1 +5 -6)] (+1 [X-6])

[SAT(+6 [X-4] [X-5])] [SAT(-1 -6 +4)] [SAT(-3 [X-2] [X-4])]

[SAT(+4 +1)] [SAT(+4)]

$6 = FALSE$ forces us to Unit-Propogate $1 = TRUE$, and

$3 = FALSE$ and $4 = TRUE$ force us to Unit-Propogate $1 = FALSE$



This is a contradiction. Backing up on our graph, the assumptions that caused this were $4 = TRUE$, and $2 = TRUE$, so we can add the clause (-2 -4).

Problem 5.2

Perform unit propagation on the following sentence. Show each step and determine all models that makes the sentence true.

$$(\neg P \vee Q \vee S), (\neg P \vee \neg R \vee S), (P), (\neg Q \vee R), (\neg S)$$

Solution:

a. Propogate $P = TRUE$:

$$(Q \vee S), (\neg R \vee S), (\neg Q \vee R), (\neg S)$$

b. Propogate $S = FALSE$:

$$(Q), (\neg R), (\neg Q \vee R)$$

c. Propogate $Q = TRUE$:

$$(\neg R), (R)$$

d. Propogate $R = FALSE$:

$$()$$

Because we came across an empty clause, we know this is unsatisfiable. Thus, there are no satisfying models.

Problem 5.3

Given a finite set S and a binary operation $* : S \times S \rightarrow S$, the tuple (S,*) is called a group if and only if the following properties hold:

- Neutral Element: $\exists n \in S \; \forall a \in S : n * a = a = a * n$
- Associative: $\forall a, b, c \in S : (a * b) * c = a * (b * c)$
- Inverse: $\forall a \in S : \exists b \in S : a * b = n = b * a$

a. Given the set $\{0, 1\}$ and an operation $* : \{0, 1\} \times \{0, 1\} \rightarrow \{0, 1\}$, express the fact that $(\{0, 1\}, *)$ is a group in propositional logic.

b. Given a refutation complete algorithm for PL, how could you use the solver to show that the neutral element of any group over $\{0, 1\}$ is unique?

c. How could you show that the inverse element is unique?

Solution:

We will label our variables X_{ijk} to denote $i * j = k$. First we need that for every set of inputs, * produces an output in $0, 1$:

$$(X_{000} \vee X_{001}) \wedge (X_{010} \vee X_{011}) \wedge (X_{100} \vee X_{101}) \wedge (X_{110} \vee X_{111})$$

Now we need that * is well defined:

$$(X_{000} \Rightarrow \neg X_{001}) \vee (X_{001} \Rightarrow \neg X_{000})(X_{100} \Rightarrow \neg X_{101}) \vee (X_{101} \Rightarrow \neg X_{100})(X_{010} \Rightarrow \neg X_{011}) \vee (X_{011} \Rightarrow \neg X_{010})(X_{110} \Rightarrow \neg X$$

Ok, so now that * is well defined we need the properties of a group, like an identity:

$$(X_{000} \wedge X_{101} \wedge X_{011}) \vee (X_{010} \wedge X_{100} \wedge X_{111})$$

Associative (I use $\bigvee$ and $\bigwedge$ to make it more readable:

$$\bigwedge_{a,b,c} [\bigvee_{ijk} (X_{abi} \wedge X_{ick} \quad \wedge \quad X_{ajk} \wedge X_{bcj})]$$

Inverse:

$$[(X_{000} \wedge X_{101} \wedge X_{011}) \wedge ((X_{000} \wedge X_{000}) \vee (X_{010} \wedge X_{100})) \wedge ((X_{110} \wedge X_{110}) \vee (X_{100} \wedge X_{010}))] \vee$$

$$[(X_{010} \wedge X_{100} \wedge X_{111}) \wedge ((X_{001} \wedge X_{001}) \vee (X_{011} \wedge X_{101})) \wedge ((X_{111} \wedge X_{111}) \vee (X_{101} \wedge X_{011}))]$$

Neutral Element unique:

$$KB \wedge \neg [X_{000} \wedge X_{101} \wedge X_{011} \wedge X_{010} \wedge X_{100} \wedge X_{111}]$$

Inverse unique:

$$(X_{000} \wedge X_{101} \wedge X_{011}) \wedge [X_{000} \wedge X_{000} \wedge (\neg [X_{010} \wedge X_{100}])] \wedge [X_{100} \wedge X_{010} \wedge (\neg [X_{110} \wedge X_{110}])] \vee$$

$$(X_{010} \wedge X_{100} \wedge X_{111}) \wedge [X_{001} \wedge X_{001} \wedge (\neg [X_{011} \wedge X_{101}])] \wedge [X_{101} \wedge X_{011} \wedge (\neg [X_{111} \wedge X_{111}])]$$

Problem 5.4

Define a consistent vocabulary and represent the following sentences in first-order logic:[1]

a. Every person who buys a policy is smart.

[1] From Russell and Norvig 8.6, e-k.

b. No person buys an expensive policy.

c. There is an agent who sells policies only to people who are not insured.

d. There is a barber who shaves all men in town who do not shave themselves.

e. A person born in the UK, each of whose parents is a UK citizen or a UK resident, is a UK citizen by birth.

f. A person born outside the UK, one of whose parents is a UK citizen by birth, is a UK citizen by descent.

g. Politicians can fool some people all of the time, and they can fool all of the people some of the time, but they can't fool all of the people all of the time.

Solution:

Vocab:

$Buys(x, y, z)$: person x buys y from z

$Sells(x, y, z)$: person x sells y to z

$Shaves(x, y)$: person x shaves person y

$Born(x, c)$: person x born in country c

$Parent(x, y)$: x is parent of y

$Citizen(x, c, r)$: x is citizen of c for reason r

$Resident(x, c)$: x lives in c

$Fools(x, y, t)$: person x fools person y at time t

$Person(x), Agent(x), Policy(x), Insured(x), Expensive(x), Smart(x), Man(x), Barber(x), Politician(x)$: x is one of these objects.

a. $\forall x Person(x) \wedge (\exists y, z Policy(y) \wedge Buys(x, y, z)) \Rightarrow Smart(x)$

b. $\forall x, y, z Person(x) \wedge Policy(y) \wedge Expensive(y) \Rightarrow \neg Buys(x, y, z)$

c. $\exists x Agent(x) \wedge \forall y, z Sells(x, y, z) \Rightarrow \neg Insured(z)$

d. $\exists x Barber(x) \wedge \forall y Man(y) \wedge \neg Shaves(y, y) \Rightarrow Shaves(x, y)$

e. $\forall p Born(p, UK) \wedge (\forall x, Parent(x, p) \Rightarrow \exists r Citizen(x, UK, r) \vee Resident(x, UK)) \Rightarrow Citizen(p, UK, Birth)$

f. $\forall p(\neg Born(p, UK) \wedge \exists x Parent(x, p) \vee Citizen(x, UK, Birth)) \Rightarrow Citizen(p, UK, Descent)$

g. $\forall x Politician(x) \Rightarrow (\exists y \forall t Fools(x, y, t)) \wedge (\exists t \forall y Fools(x, y, t)) \wedge \neg(\forall y \forall t Fools(x, y, t))$