Mathematical Tools

Sets

- order doesn't matter
- finite: $\{2, 4, 6\}$
- infinite: $\{2, 4, 6, \ldots\}$, $\{x \mid x \text{is even}\}$
- subset: $U \subset V$, $U \subseteq V$
- union: $U \cup V$
- intersection: $U \cap V$

Sequences

- order matters
- $(1, 3, 5, \ldots)$
- finite sequence called a *tuple*
- 2 elements is *pair*
- k elements is k-tuple
- *powerset* of U is set of all subsets
- Cartesian product of $U \times V$

Mathematical Tools

A *function* or *mapping*

$$f : X \to Y$$

- : *domain* $\to$ *range*
- input values are called *arguments*.
- A k-ary function has k arguments.
- A *predicate* is a function with range {*true*, *false*}.
- A *relation* is function whose domain is a k-tuple.

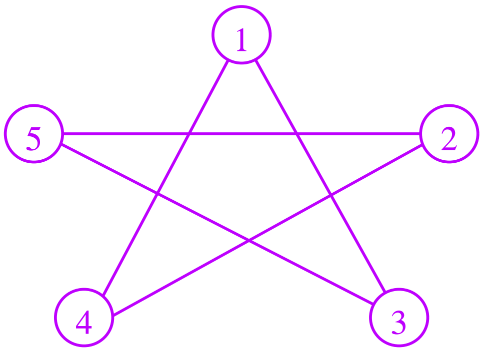
The "SCISSORS, PAPER, ROCK" game as a predicate:

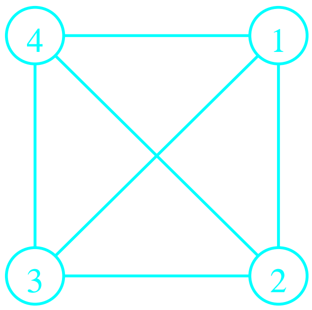
	SCISSORS	PAPER	ROCK
SCISSORS	*false*	*true*	*false*
PAPER	*false*	*false*	*true*
ROCK	*true*	*false*	*false*

and as a relation:

{(SCISSORS, PAPER), (PAPER, ROCK), (ROCK, SCISSORS)} .

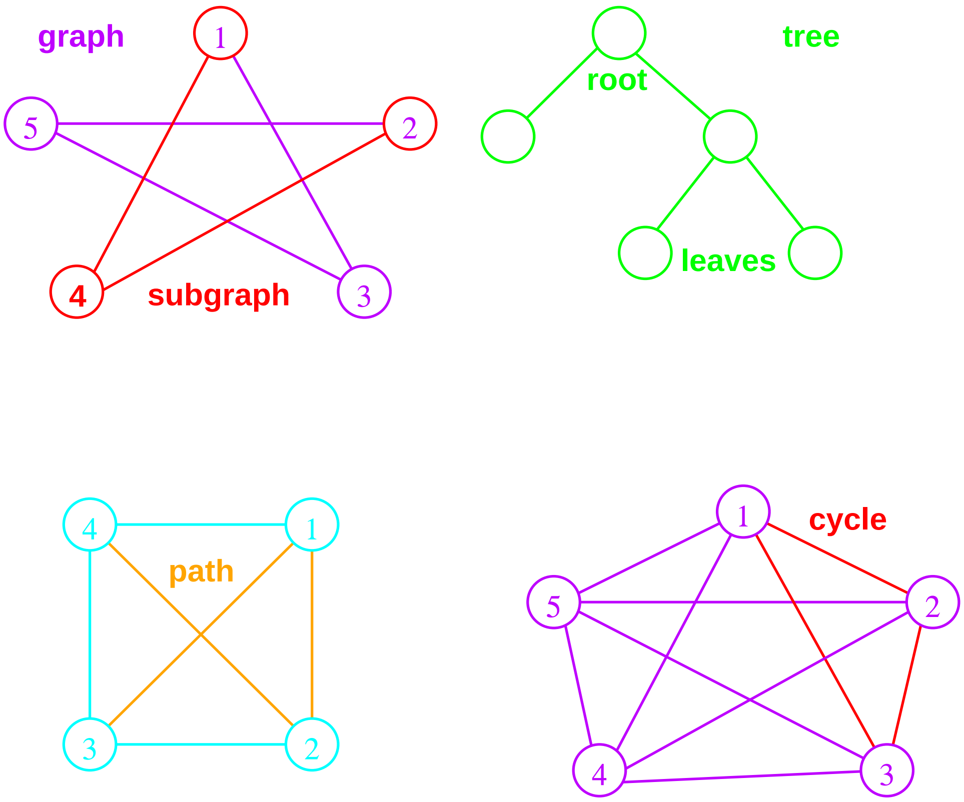
Graphs



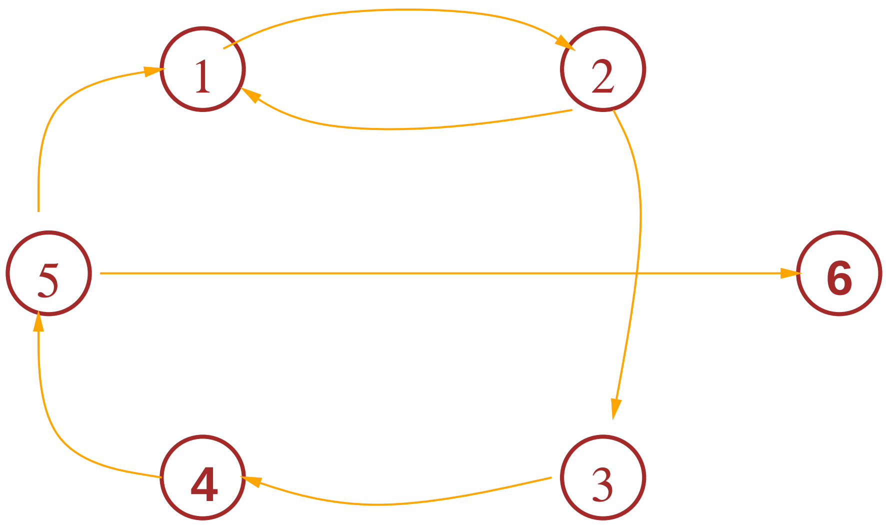


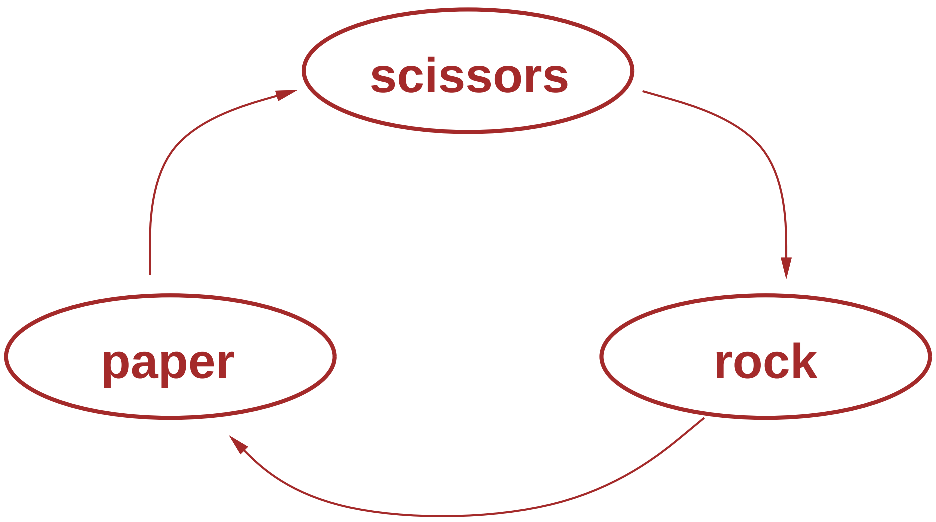
- $G = (V, E)$, where
- V is set of *nodes* or *vertices*, and
- E is set of *edges*
- *degree* of a vertex is number of edges

Graphs

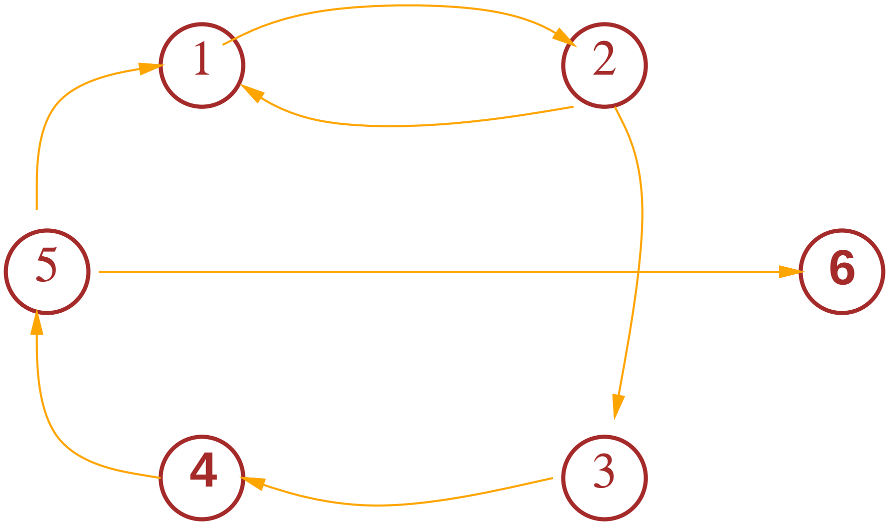

graph
1
5
2
4
subgraph
3
tree
root
leaves
4
1
path
3
2
1
cycle
5
2
4
3

Directed Graphs





Adjacency Matrix



$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Strings and Languages

- an *alphabet* is a finite set of *symbols*
- a *string over an alphabet* is a finite sequence of symbols from that alphabet.
- the *length* of a string is the number of symbols
- the *empty string* ε
- *reverse*: $abcd$ is $dcba$.
- *substring*: xyz in $xyzzy$.
- *concatenation* of xyz and zy is $xyzzy$.
- x^k and $x \cdots x$, k times.
- a *language* is a set of strings.

Proofs

We will use the following basic kinds of proofs.

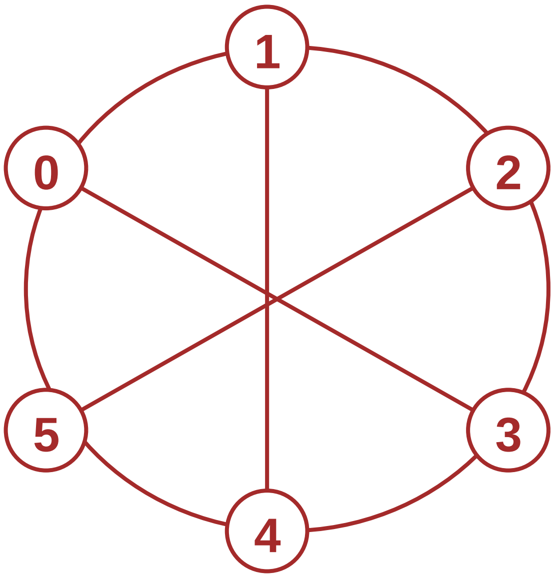
- by construction
- by contradiction
- by induction
- by reduction

Sometimes we mix them.

Proof by Construction

A graph is k-regular if every node has degree k.

Theorem: for every even $n > 2$, there exists a 3-regular graph with n nodes.



Proof: Construct $G = (V, E)$, where $V = \{0, 1, \ldots, n-1\}$ and

$$
\begin{aligned}
E &= \{\{i, i+1\} \mid \textit{for } 0 \le i \le n-2\} \cup \{n-1, 0\} \\
&\quad \cup \{\{i, i+n/2\} \mid \textit{for } 0 \le i \le n/2 - 1\}\,.
\end{aligned}
$$

Note: picture is helpful, but *not* a proof!

Proof by Contradiction

Theorem: $\sqrt{2}$ is irrational.

Proof: Suppose not. Then $\sqrt{2} = \frac{m}{n}$, where m and n are relatively prime.

$$\begin{aligned} n\sqrt{2} &= m \\ 2n^2 &= m^2 \end{aligned}$$

So m^2 is even, and so is $m = 2k$.

$$\begin{aligned} 2n^2 &= (2k)^2 \\ &= 4k^2 \\ n^2 &= 2k^2 \end{aligned}$$

So n^2 is even, and so is n.

Thus both m and n are even, not relatively prime!

Reductio ad absurdam.

Induction

Prove properties of elements of an infinite set.

$$\mathcal{N} = \{1, 2, 3, \ldots\}$$

To prove that $\wp$ holds for each element, show:

- *base step:* show that $\wp(1)$ is true.
- induction step: show that if $\wp(i)$ is true (the induction hypothesis), then so is $\wp(i+1)$.

Induction Example

Theorem: All cows are the same color.

Base step A single-cow set is definitely the same color.

Induction Step: Assume all sets of i cows are the same color. Divide the set $\{1, \ldots, i+1\}$ into $U = \{1, \ldots, i\}$, and $V = \{2, \ldots, i+1\}$.

- all cows in U are the same color by the induction hypothesis.
- all cows in V are the same color by the induction hypothesis.
- all cows in $U \cap V$ are the same color by the induction hypothesis.

Ergo, all cows are the same color.

Quod Erat Demonstrandum.