Games

A *game* is

- a competition between two parties
- each attempting to reach a goal
- by following specified rules

Examples

- chess
- war games
- class conflict

Games

Games are closely related to quantifiers.

- each quantified statement has a corresponding game
- each game has a corresponding quantified statement.

Helpful because

- games give insight into formula structure
- formulas give insight into game complexity

The Formula Game

Let

$$\phi = \exists x_1 \, \forall x_2 \, \exists x_3 \, \ldots Q x_k [\psi]$$

be

- a quantified Boolean formula
- in prenex normal form,
- and Q is either $\exists$ or $\forall$

The Formula Game

$$\phi = \exists x_1 \, \forall x_2 \, \exists x_3 \, \ldots Q x_k [\psi]$$

The game:

- Two players, called A and E
- take turns selecting values for $x_1, \ldots, x_k$
- A selects values for variables bound to $\forall$
- E selects values for variables bound to $\exists$
- order of play is order of quantifiers
- E wins if ψ is *true* at the end
- A wins if ψ is now *false*

The Formula Game

$$\phi_1 = \exists x_1 \, \forall x_2 \, \exists x_3 \; [(x_1 \vee x_2) \wedge (x_2 \vee x_3) \wedge (\overline{x_2} \vee \overline{x_3})] \, .$$

- Player E picks x_1
- Player A picks x_2
- Player E picks x_3

The Formula Game

$$\phi_1 = \exists x_1 \, \forall x_2 \, \exists x_3 \; [(x_1 \vee x_2) \wedge (x_2 \vee x_3) \wedge (\overline{x_2} \vee \overline{x_3})] \, .$$

- Player E picks $x_1 = 1$
- Player A picks $x_2 = 0$
- Player E picks $x_3 = 1$

$$\Big[(1 \vee 0) \wedge (0 \vee 1) \wedge (\overline{0} \vee \overline{1})\Big] = 1.$$

So player E wins.

The Formula Game

$$\phi_2 = \exists x_1 \, \forall x_2 \, \exists x_3 \; [(x_1 \vee x_2) \wedge (x_2 \vee x_3) \wedge (x_2 \vee \overline{x_3})] \, .$$

Now, player A can always win:

- no matter what E picks for x_1,
- Player A picks $x_2 = 0$
- Player E's last move irrelevant.

A has a winning strategy.

The Formula Game

FORMULA-GAME $=$

$\{\langle\phi\rangle | E$ has a winning strategy in ϕ's game$\}$

Question: What does this have to do with complexity?

The Formula Game

FORMULA-GAME $=$

$\{\langle\phi\rangle | E$ has a winning strategy in ϕ's game$\}$

Theorem: FORMULA-GAME is PSPACE-complete.

The Formula Game

Theorem: FORMULA-GAME is PSPACE-complete.

Idea: FORMULA-GAME is exactly TQBF.

The Formula Game

Theorem: FORMULA-GAME is PSPACE-complete.

Proof:

$$\phi = \exists x_1 \, \forall x_2 \, \exists x_3 \; \ldots [\psi]$$

is true when

- a setting for x_1 exists such that
- for any setting of x_2,
- a setting for x_3 exists such that
- ...
- ψ is true under these settings.

Thus $\phi \in$ TQBF if and only if $\phi \in$ FORMULA-GAME.

Formula Game

Reduction holds even when formula does not simply alternate $\exists$ and $\forall$.

$$\phi = \forall x_1, x_2, x_3 \, \exists x_4, x_5 \, \forall x_6 [\psi]$$

- Player A picks x_1, x_2, x_3
- Player B picks x_4, x_5
- Player A picks x_6

Same analysis holds.

Now that we know the formula game is PSPACE-complete, we can analyze other games.

The Geography Game

Good for a long car trip with kids.

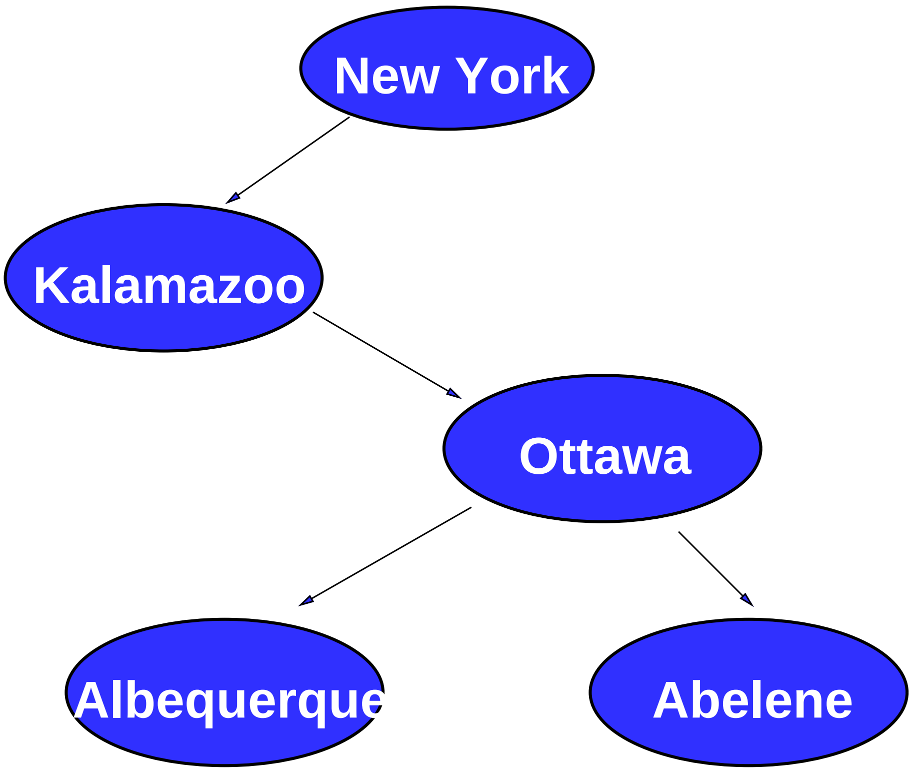
- players alternate naming cities
- designated starting city (Pawtucket?)
- each city named must begin with the last letter of previous city
- no repetitions

For example

- New York
- Kalamazoo
- Ottowa
- Abelene . . .

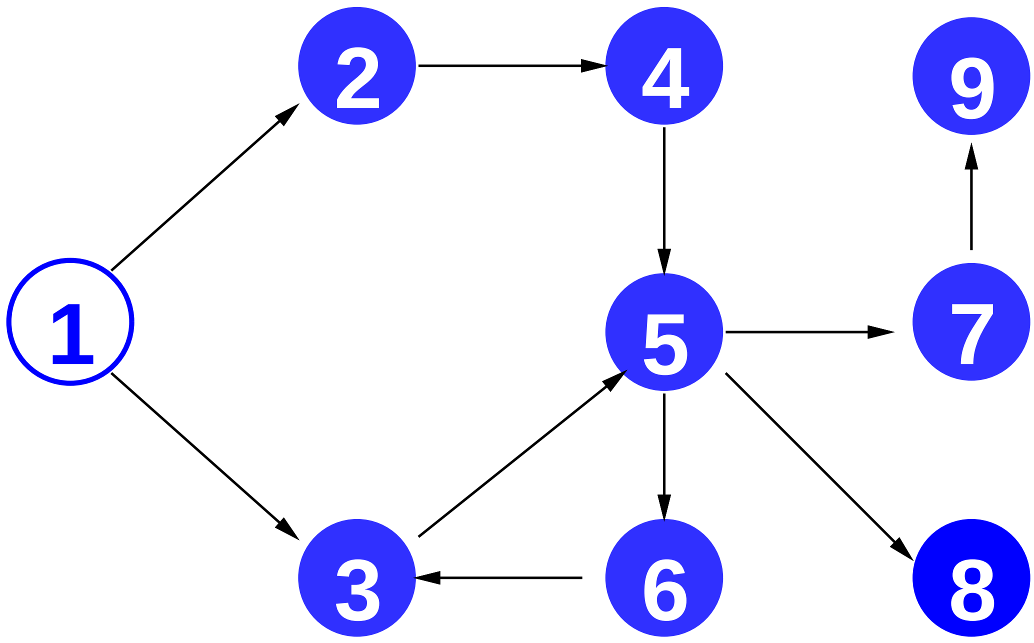
The Geography Game

We model this game as a directed graph.



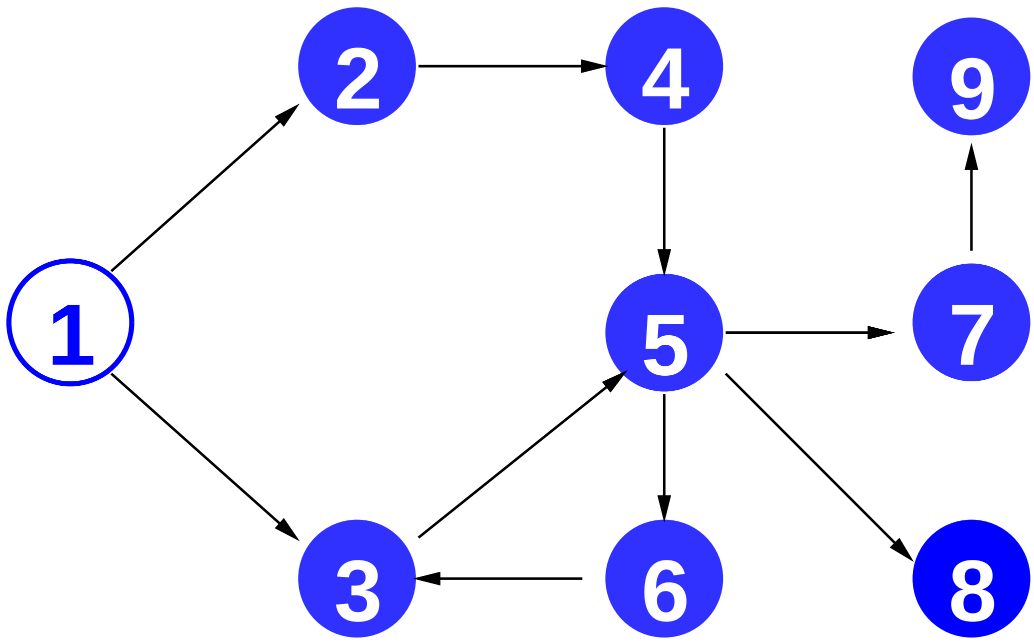
- players alternate edges
- construct simple path
- first one unable to extend the path loses

Generalized Geography Game



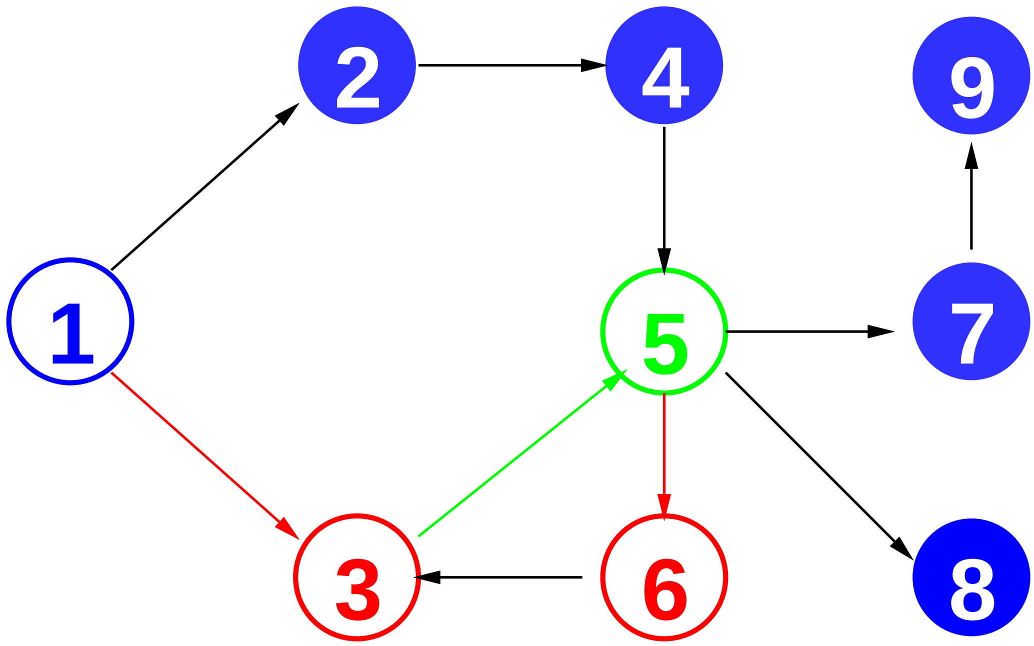
- arbitrary directed graph
- designated starting node
- players alternate edges
- construct simple path
- first one unable to extend the path loses

Generalized Geography Game



Claim: Player A has a winning strategy.

Generalized Geography Game

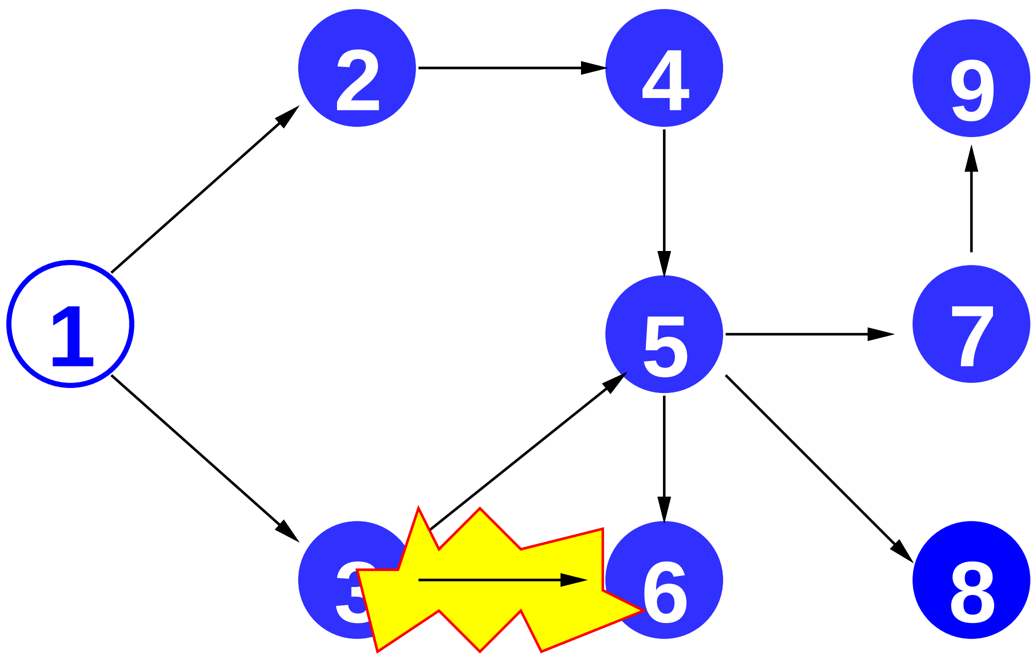


Claim: Player A has a winning strategy.

- A chooses 3.
- B must choose 5.
- A chooses 6

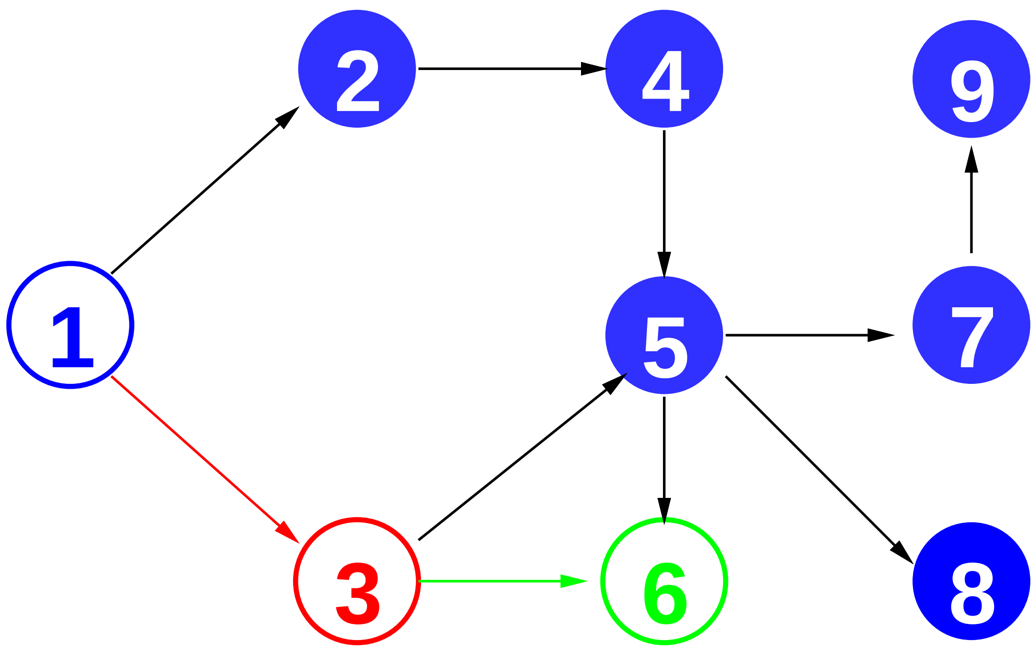
And B loses because it has no place to go.

Generalized Geography Game



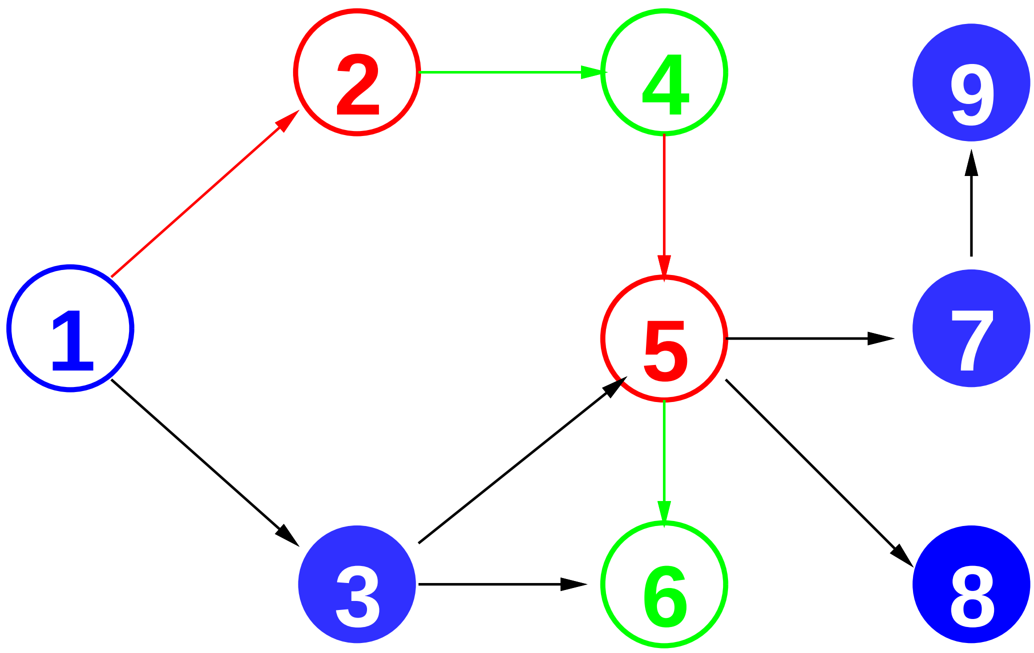
Claim: If we reverse one edge, then B has a winning strategy.

Generalized Geography Game



- if A starts with 3, as before,
- then B responds with 6 and wins immediately.

Generalized Geography Game



- so A must start with 2
- B responds with 4
- A forced to 5
- B wins with 6

Generalized Geography Game

GG $=$

$\{\langle G, b\rangle |$First player has a winning strategy in G.G. game on graph G starting at $b\}$

Theorem: GG is PSPACE-complete.

Generalized Geography Game: Idea

To show that GG $\in$ PSPACE,

- we show a recursive algorithm similar to the one used for TQBF,
- runing in polynomial space.

To show that GG is PSPACE-hard

- poly-time reduction from FORMULA-GAME to GG
- convert formula game to Generalized Geography game
- play on graph mimics formula play.

Generalized Geography Game: Proof

M: on input $\langle G, b\rangle$ where G is a digraph and b a node of G.

- Remove b and all arrows touching it to get G_1.
- For each $b_1, b_2, \ldots, b_k$ to which b originally pointed, recursively call M on $\langle G_1, b_i\rangle$.
- If any of these accept, the second player has a winning strategy, so *reject*.
- Otherwise the first player must have a winning strategy, so *accept*.

Generalized Geography Game: Proof

This algorithm clearly decides GG.

What about space?

- each level of recursion adds a single node to the stack,
- at most m levels occur, where m is number of vertexes
- so algorithm runs in linear space.

Generalized Geography Game: Proof

Show that FORMULA-GAME $\leq_p$ GG.

Map formula

$$\phi = \exists x_1 \, \forall x_2 \, \exists x_3 \ldots \exists x_k \, [\psi]$$

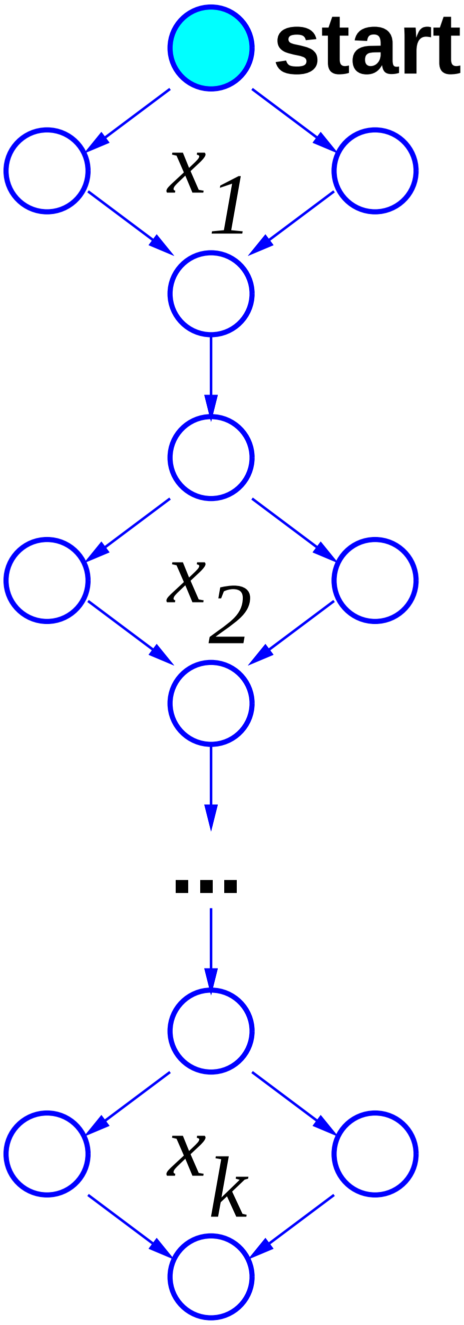
into an instance of GG.

Assume (for simplicity)

- ψ is in conjunctive normal form with clauses $c_1, \ldots, c_\ell$.
- quantifiers begin and end with $\exists$
- quantifiers alternate $\exists$ and $\forall$

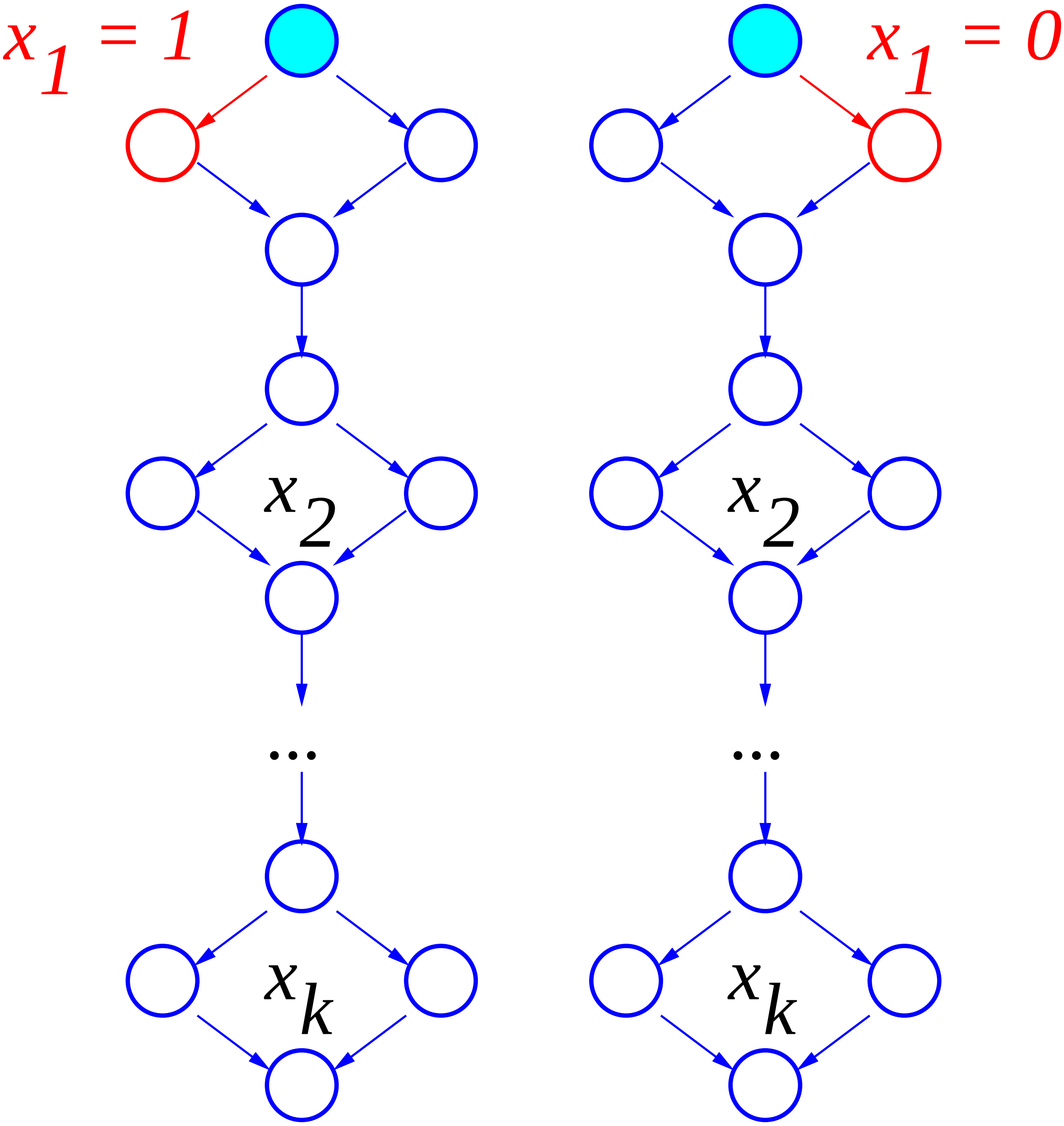
A non-conforming formula can be fixed by adding dummy variables.

Generalized Geography Game: Proof



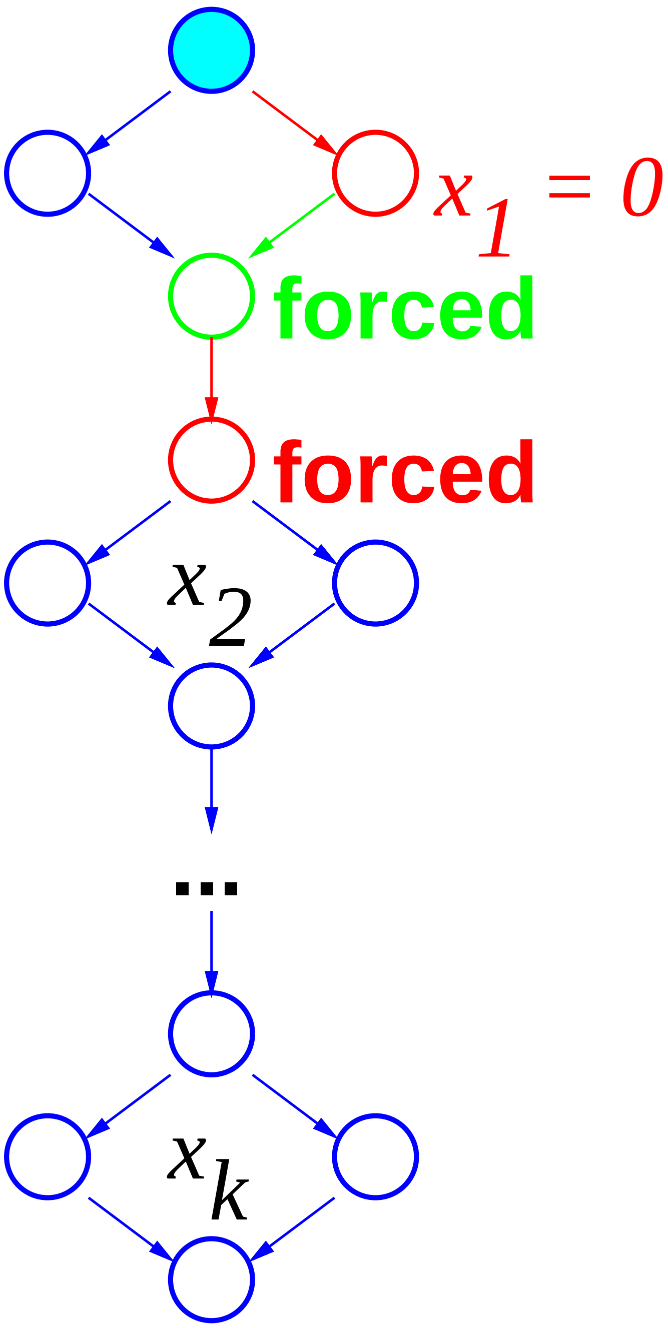
This is only part of the full graph!

Generalized Geography Game: Proof



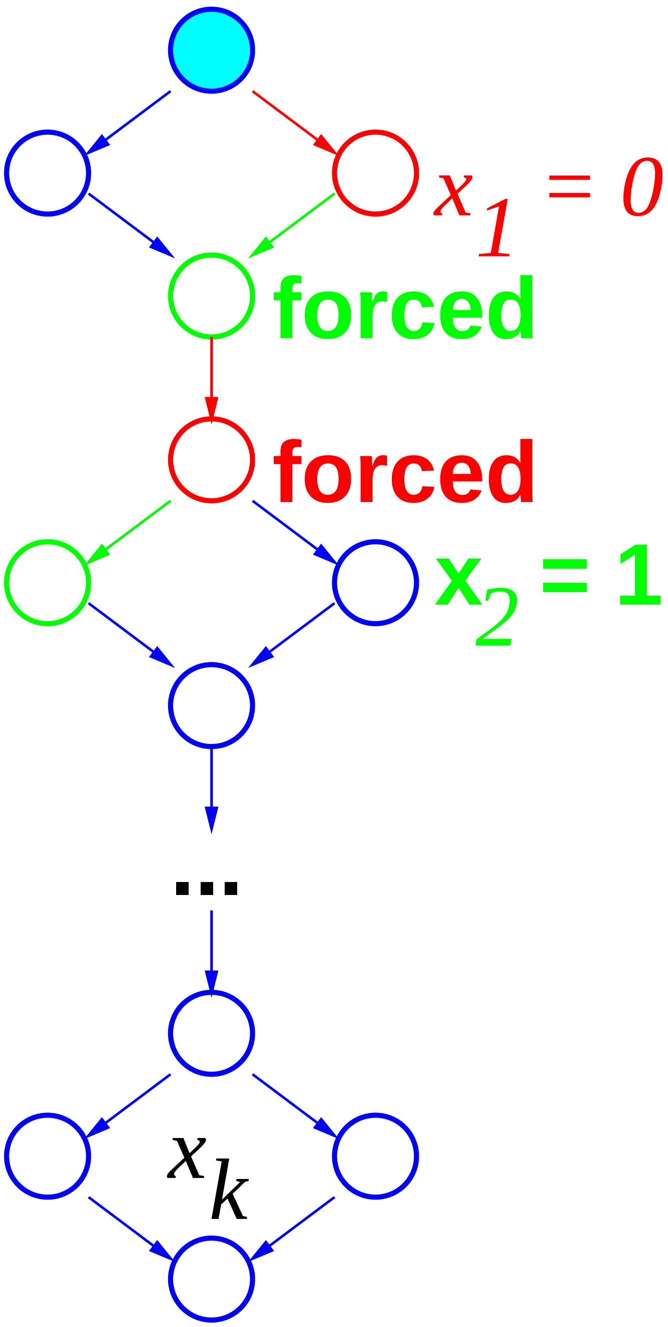
- A goes left if $x_1 = 1$
- A goes right if $x_1 = 0$

Generalized Geography Game: Proof



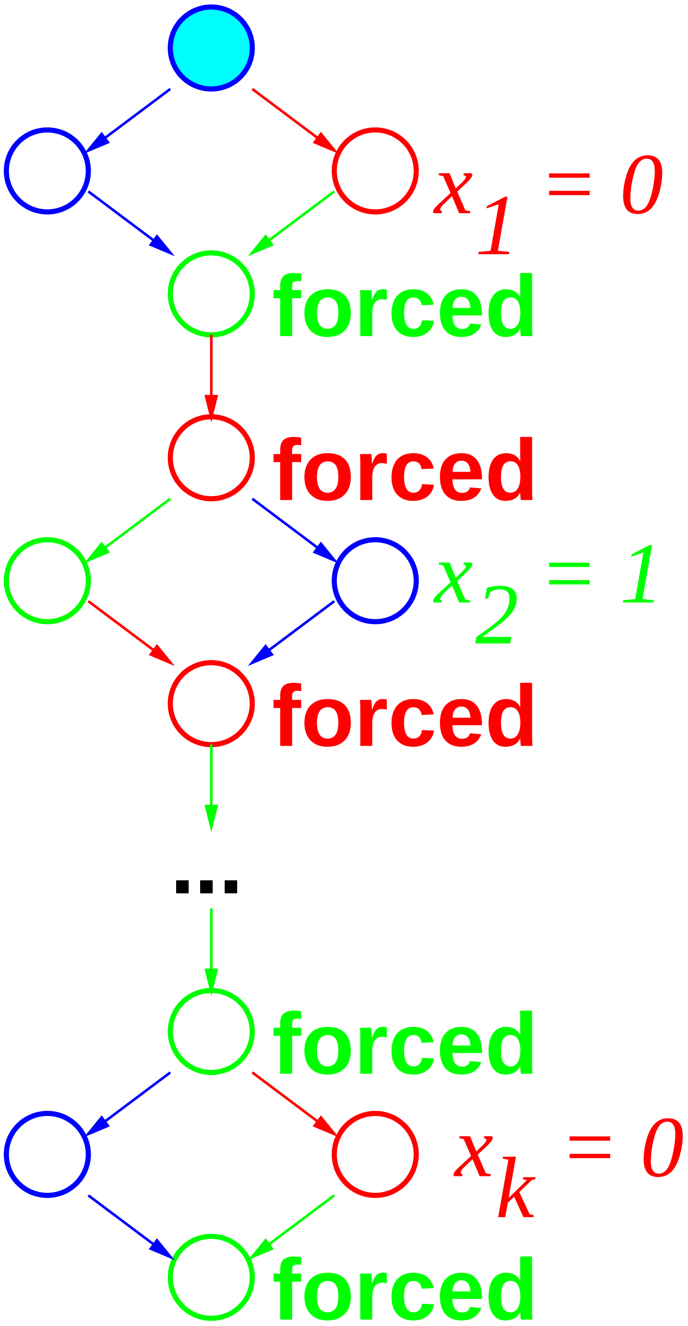
- B's move is forced
- A's move is forced

Generalized Geography Game: Proof



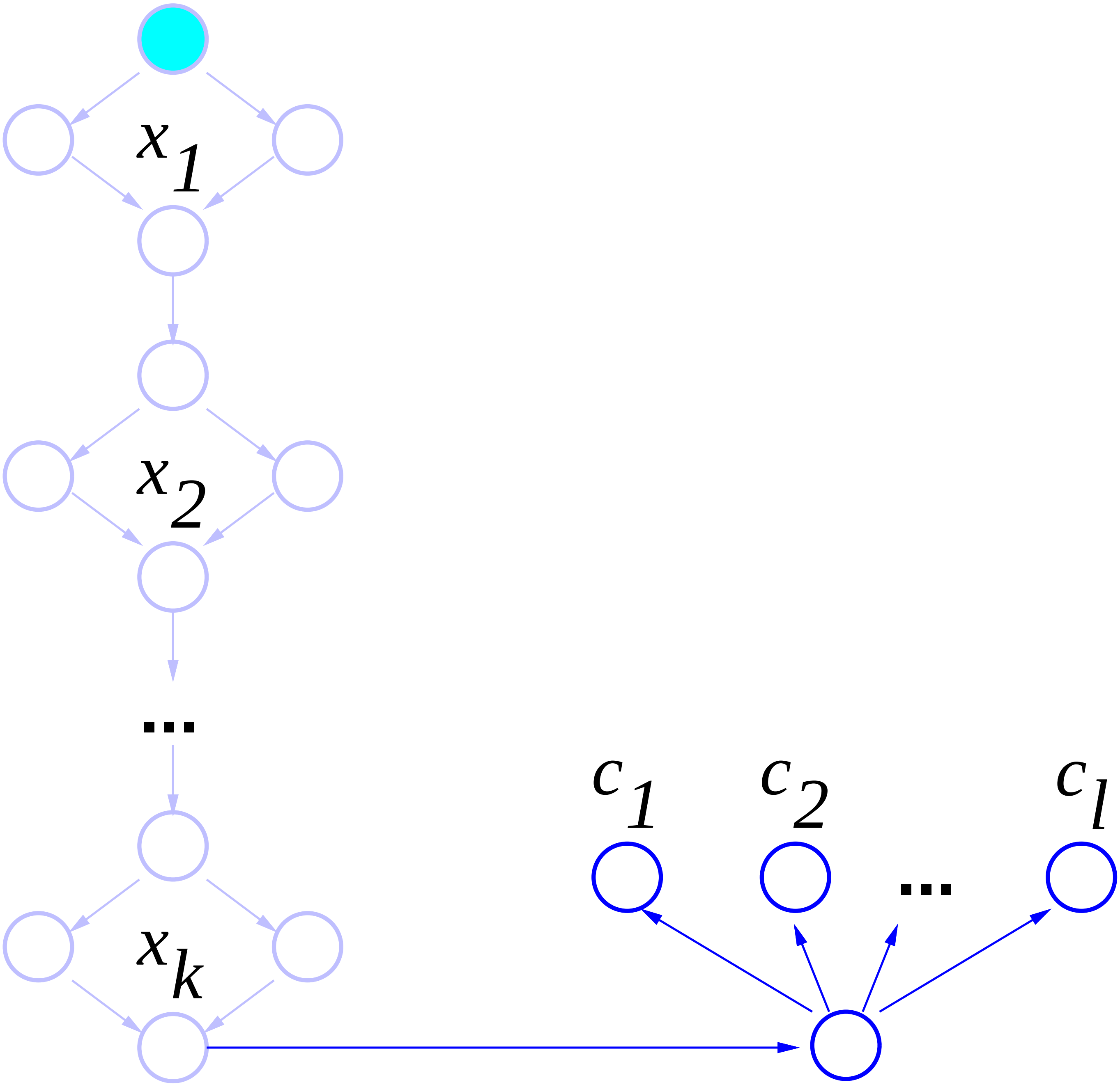
- B goes left if $x_2 = 1$
- B goes right if $x_2 = 0$

Generalized Geography Game: Proof



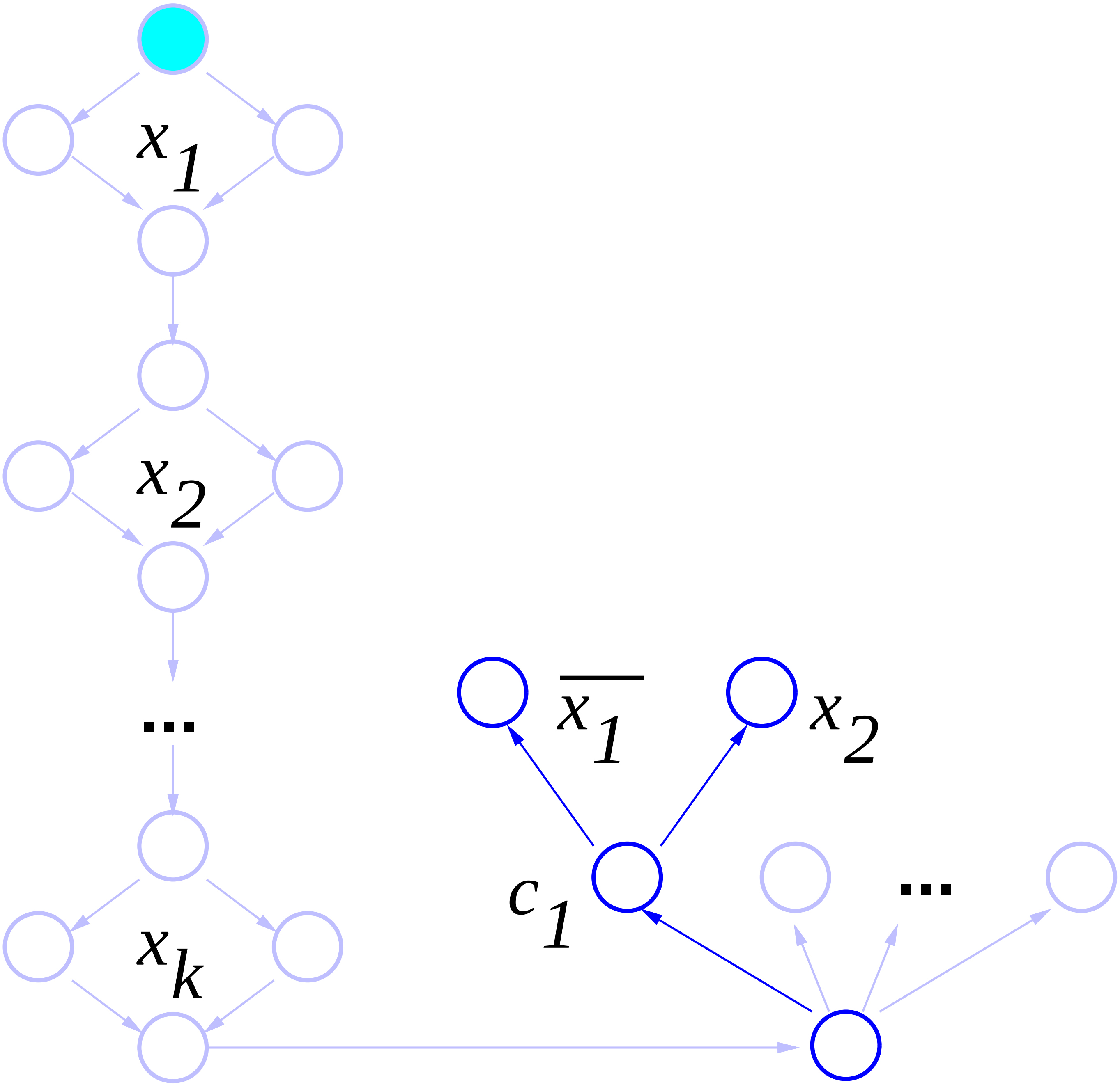
Eventually, each side has assigned a value to its variables.

Generalized Geography Game: Proof



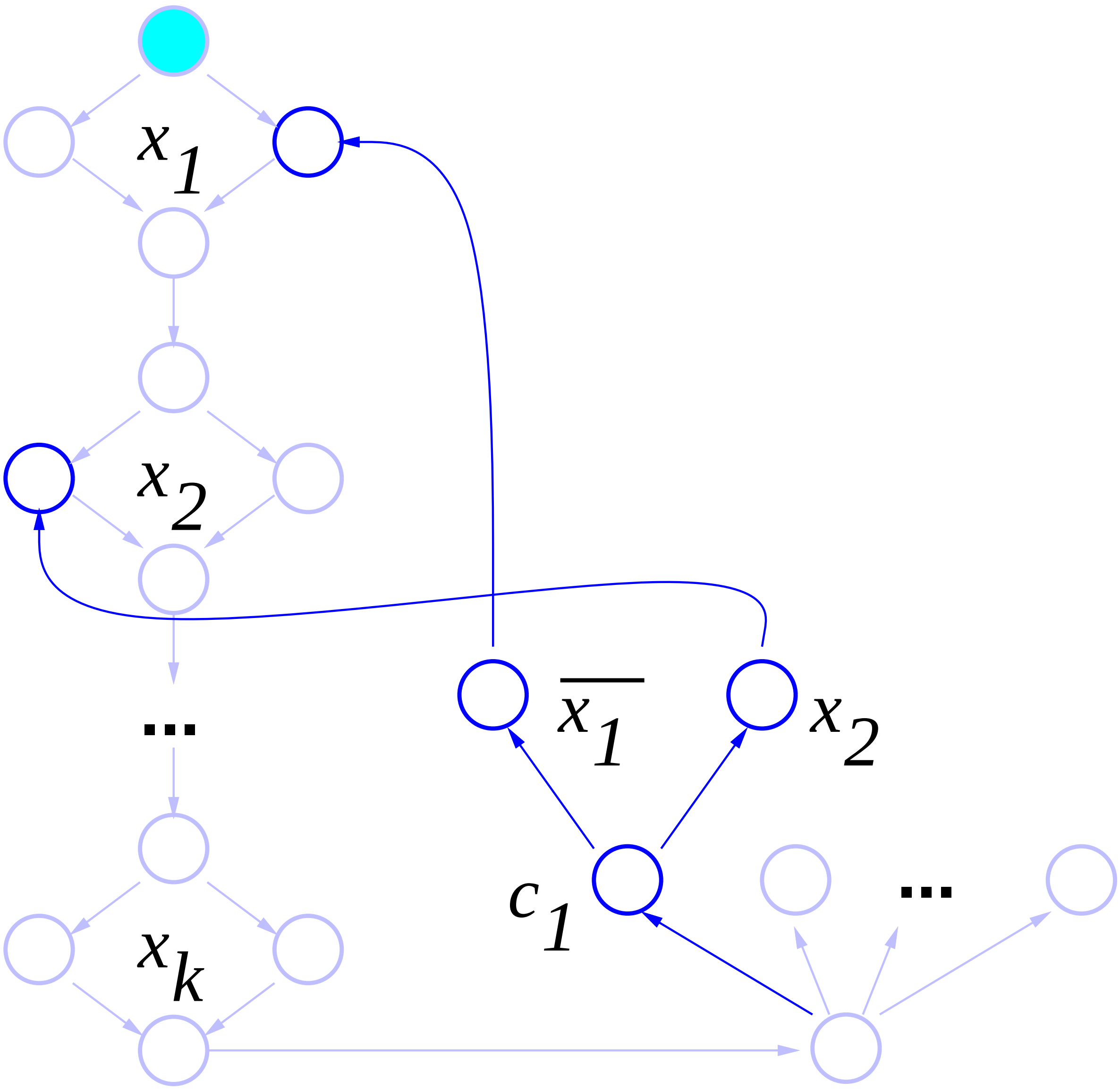
Add vertex for each clause.

Generalized Geography Game: Proof



Add vertex for each literal in each clause.

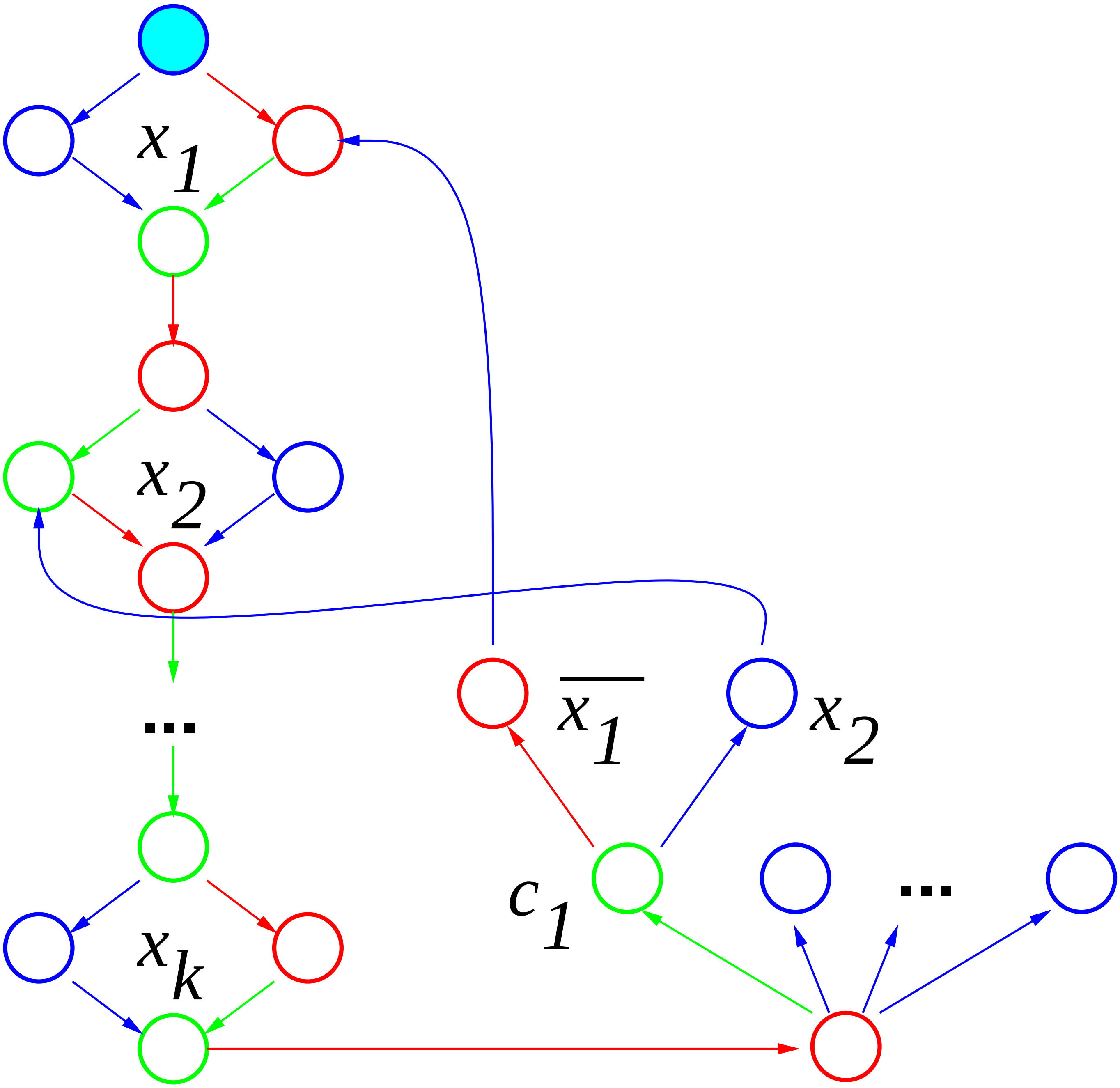
Generalized Geography Game: Proof



Link each literal vertex of clause to

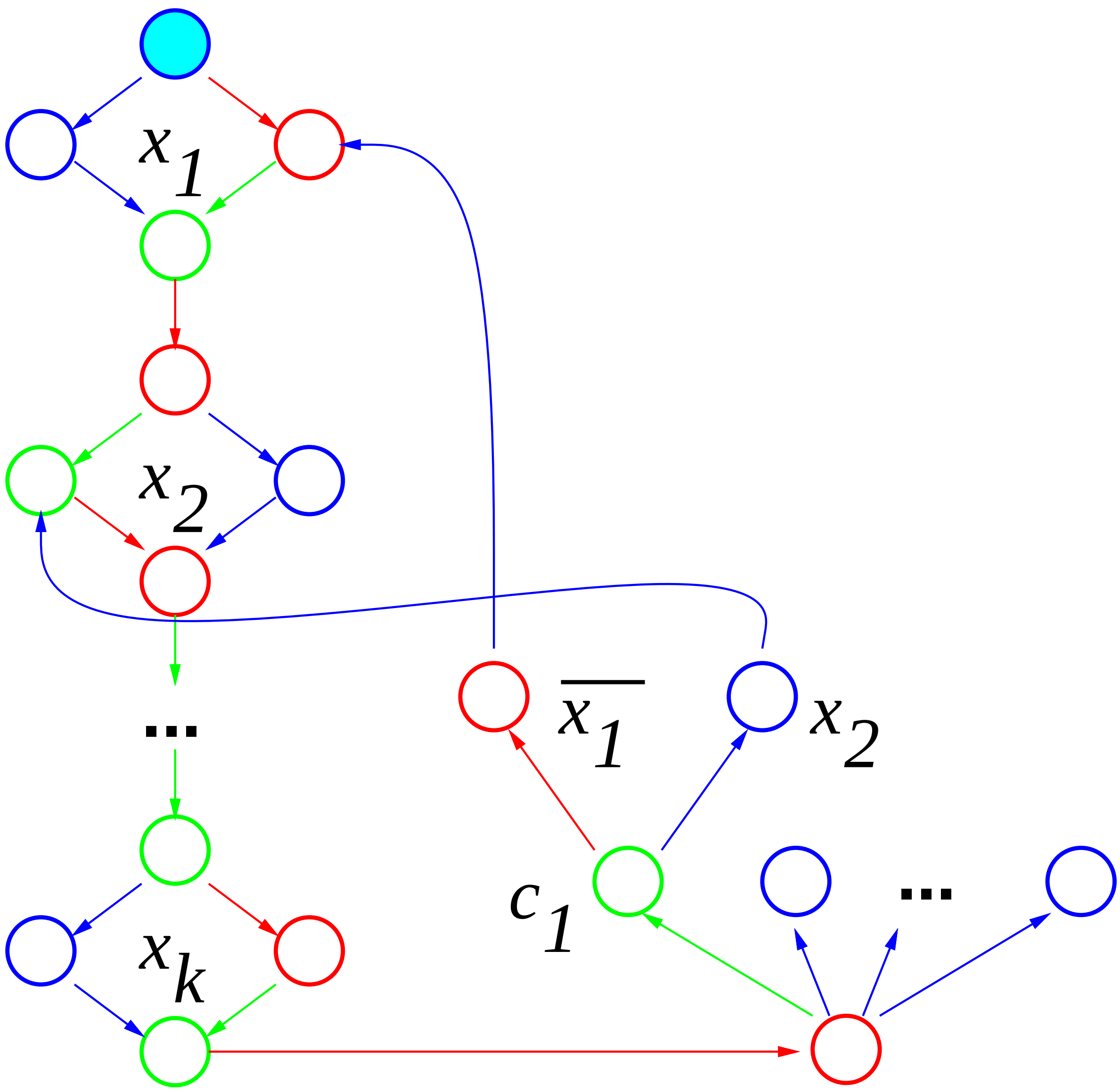
- LHS of x_i if x_i in c_js.
- RHS of x_i if $\overline{x_i}$ in c_js.

Generalized Geography Game: Proof



- A's move is forced.
- B chooses a clause
- A chooses a literal

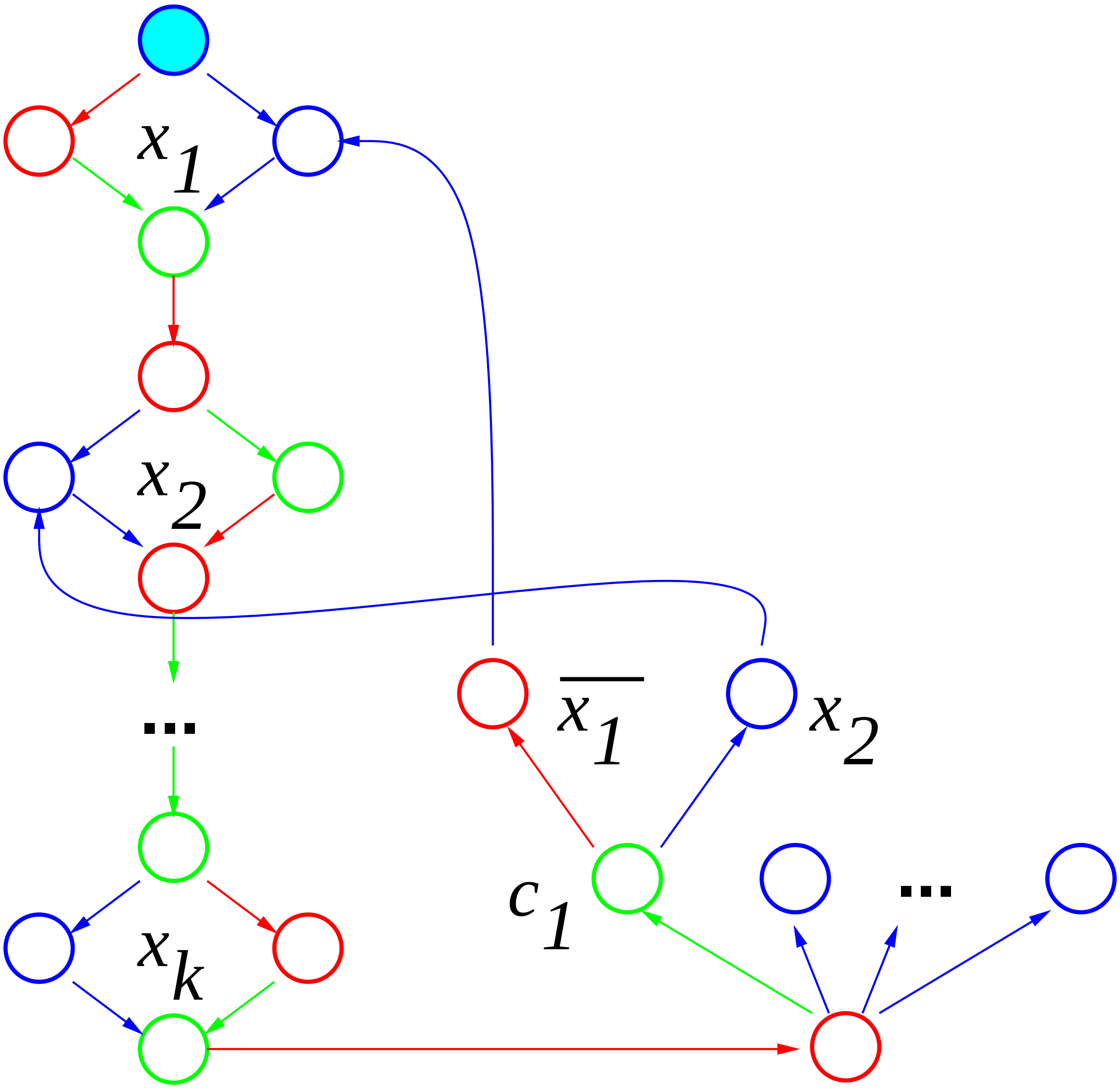
Generalized Geography Game: Proof



A wins if and only if

- its literal is assigned 1
- every clause has a true literal
- formula is true

Generalized Geography Game: Proof



B wins if and only if

- all literals are 0
- some clause's literals are all 0
- formula is false

Remark

The problem of deciding who has a win from an arbitrary position in an $n \times n$ board in Go or Checkers is PSPACE-hard.

It is PSPACE-complete if rules for draw are appropriate to prevent exponentially long games.