Problem Set 5

CS157 - Spring 2009

Due: Tuesday, April 21, 2009 10:30am

Problem 1: Unique Minimum Cut

(a) Suppose you are given a directed graph $G = (V, E)$ with integer edge capacities, and you are also given a maximum s-t flow f on G (where $s, t \in V$). Say we pick some edge $e \in E$ and increase its capacity by 1. Give an algorithm that finds a maximum s-t flow on the modified graph in time $O(|E| + |V|)$.

(b) Give an algorithm to determine whether a graph $G = (V, E)$ with nonnegative edge capacities has a *unique* minimum s-t cut.

Problem 2: Scheduling

Consider the problem of scheduling weekly TA office hours. There are k non-overlapping one-hour slots $I_1, I_2, \ldots, I_k$ throughout the week during which n TAs may hold office hours, and each TA is available during a subset of those slots. The professor wants to make sure that no single TA holds more than b office hours per week (to prevent fatigue) and that in total the entire TA staff holds exactly c office hours per week. Each TA should only be assigned slots during which he or she is available. There should be only one TA assigned to each slot.

(a) Give an efficient algorithm for assigning a subset of the slots to the TAs (a *schedule*) such that the above constraints are satisfied. If the constraints cannot be satified, the algorithm should correctly report this fact.

(b) Repeat the task from part (a), with the additional constraint that each TA must hold at least a office hours per week.

(c) Repeat the task from part (b), with the additional constraint that there must be at least d_i office hours held on the ith day of the week.

Problem 3: Path covering

(a) A *path cover* of a directed acyclic graph $G = (V, E)$ is a set P of paths such that every vertex in V is included in exactly one path in P. Paths may be of any length, including zero. (A zero-length path consists of a single vertex and no edges.) A *minimum path cover* of G is a path cover containing the fewest possible paths. Give an efficient algorithm to find a minimum path cover of a directed acyclic graph $G = (V, E)$. (*Hint:* Reduce this to a maximum flow problem.)

(b) Let S denote a set of objects $\{x_1, x_2, \ldots, x_n\}$, and let $S_1, S_2, \ldots, S_k$ denote k distinct subsets of S. Consider a datastructure called a 'blob' which is an sequence of some subset of the elements of S. A blob B *covers* subset S_i if a prefix of B is identical to some ordering of S_i, where a prefix is a contiguous subsequence of B starting with the first element of B. Write an algorithm that, given the subsets $S_1, \ldots, S_k$ and a number $\ell < k$, decides whether there exists a collection of ℓ blobs such that every subset is covered. If such a collection exists, the algorithm returns. (Certainly for $\ell = k$, the answer is yes: for each subset S_i construct a blob containing the elements of S_i in arbitrary order. For $\ell < k$, the answer is yes only if at least one blob covers multiple subsets.) (*Hint:* Use your solution to part (a), and find a natural interpretation of sets as vertices of a directed acyclic graph.)

Problem 4: Linear Programming

A producer is seeking actors and investors for his new movie. There are n available actors; actor i charges s_i dollars. For funding, there are m available investors. Investor j will provide p_j dollars, but only on the condition that certain actors $L_j \subseteq \{1, 2, \ldots, n\}$ are included in the case (all of these actors must be chosen in order to receive funding from investor j). The producer's profit is the sum of the payments from investors minus the payments to actors. The goal is to maximize this profit.

(a) Express this problem as an integer linear program in which the variables take on values $\{0, 1\}$.

(b) Now relax this to a linear program in which variables take on values in the interval $[0, 1]$, and show that there must in fact be an integral optimal solution (just like with maximum flow).