

CS 159: Homework 2  
Professor John Savage

**Assigned:** February 11, 2009,  
**Due:** February 18, 2009

### Question 1

- A)** In class we defined 3SAT to be an instance of SATISFIABILITY with 3 or fewer literals per clause. Show that 3SAT\*, satisfiability with exactly 3 literals per clause, is also NP-complete.
- B)** It turns out that 2SAT is in P (see Problem 2), but MAX2SAT is NP-complete. An instance of MAX2SAT is a sequence of one- and two-literal clauses followed by an integer  $k$ . An instance is in MAX2SAT if and only if there exists an assignment of values to the variables such that at least  $k$  of the clauses are satisfied.

Show that Max2SAT is NP-complete by reducing 3SAT to it. To do this, convert each 3 variable clause to a sequence of 2 and 1 variable clauses. Use the following observation:

A satisfying assignment to  $(x \vee y \vee z)$  can satisfy at least 7 of the following clauses:

$(x), (y), (z), (w), (x \vee \bar{w}), (y \vee \bar{w}), (z \vee \bar{w}), (\bar{x} \vee \bar{y}), (\bar{y} \vee \bar{z}), (\bar{z} \vee \bar{x})$

but a non-satisfying assignment satisfies at most 6.

### Question 2

An instance of 2SAT is an instance of SATISFIABILITY with only 2 literals per clause. To show 2SAT is in P, it can be reduced to an instance of REACHABILITY.

Given an instance of 2SAT, construct a directed graph  $G(V, E)$  with a vertex  $v_\alpha$  for each literal  $\alpha$ . Notice that the clause  $(\alpha \vee \beta)$  is equivalent to  $(\bar{\alpha} \Rightarrow \beta) \wedge (\bar{\beta} \Rightarrow \alpha)$  where  $\Rightarrow$  denotes implication. For each implication, construct a directed edge  $G$ . Thus, the clause  $(\alpha \vee \beta)$  results in the introduction of two directed edges, namely,  $(v_{\bar{\alpha}}, v_\beta)$  and  $(v_{\bar{\beta}}, v_\alpha)$ .

- A)** Show that the original instance of 2SAT is not satisfiable if for some  $\alpha$  there exists a directed path from  $v_\alpha$  to  $v_{\bar{\alpha}}$  and from  $v_{\bar{\alpha}}$  to  $v_\alpha$ .
- B)** If there is no such path, show that the vertices of  $G$  can be labeled “true” and “false” such that the original instance of 2SAT is satisfied.

### Question 3

In class we highlighted the connection between NP, PSPACE, puzzles and games. Asking someone to determine if  $\mathbf{x} \in SAT$ , is similar to asking someone to solve a puzzle. Finding the solution to a puzzle may be challenging, but once solved, its solution is easy to verify.

Determining membership in a PSPACE-complete language, such as TQBF, appears more difficult. Asking whether  $\mathbf{x} \in TQBF$  is similar to asking whether a certain game has a winning strategy for one of its players. To see why, consider the following two-player game:

Initially players are presented with the boolean formula,  $\phi(\mathbf{x})$ . On the first turn, player 1 decides whether variable  $x_1$  is 1 or 0. On the second turn player 2 decides whether variable  $x_2$  is 1 or 0. Players alternate in this fashion until all variables have been assigned values. Player 1 wins if  $\phi(\mathbf{x}) = 1$ , player 2 wins if  $\phi(\mathbf{x}) = 0$ .

Now consider the language *BGAME* defined as follows. A boolean formula  $\phi(\mathbf{x}) \in BGAME$  if and only if there is a winning strategy for player 1 when  $\phi(\mathbf{x})$  is used in the game described above. Here a “winning strategy” means that there is a way for player 1 to ensure victory regardless of the choices made by player 2.

Show that *BGAME* is PSPACE-complete by giving a reduction from *TQBF* to *BGAME*. Note that in an instance of *TQBF* there may be more than one variable per quantifier.