CSCI 1590
Intro to Computational Complexity

Complement Classes and the Polynomial Time Hierarchy

John E. Savage

Brown University

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Overview

- We introduce classes of language complements and define the polynomial time hierarchy.
- We show that all languages in this hierarchy are contained in **PSPACE**.
- We introduce TQBF (totally quantified Boolean formulas). Later we show that it is **PSPACE**-complete.

Important Complexity Classes

- Time classes: **P**, **NP**, **EXPTIME**, **NEXPTIME**
- Space classes: **L**, **NL**, $\mathbf{L}^2$, **PSPACE**, **NPSPACE**.

Complements of Decision Problems

SAT
Instance: Literals $X = \{x_1, \overline{x}_1, x_2, \overline{x}_2, \ldots, x_n, \overline{x}_n\}$, and clauses $C = (c_1, c_2, \ldots, c_m)$ where each clause c_i is a subset of X.
Answer: "Yes" if for some assignment of Booleans to variables in $\{x_1, x_2, \ldots, x_n\}$, at least one literal in each clause has value 1.

Definition

The complement of a decision problem L, denoted co$L$, is the set of "No" instances of a decision problem.

Note

$\text{co}L \neq \bar{L} = \Sigma^* - L$. In fact, $L \cup \text{co}L = WF_L \subset \Sigma^*$ where WF_L is the set of well-formed strings describing "Yes" and "No" instances. That is, $\text{co}L = WF_L - L$.

Can recognize in PTIME whether a string is in WF_L.

Complements of Complexity Classes

Definition

The **complement of a complexity class** is the set of complements of languages in the class.

Example

co**NP** is set of languages consisting of “No” instances of **NP** languages.

Theorem

If $C_1 \subseteq C_2$, $coC_1 \subseteq coC_2$. If $C_1 = C_2$, $coC_1 = coC_2$

Proof.

If $\text{co}L \in \text{co}C_1$, $L \in C_1$. Thus, $L \in C_2$. This implies that $\text{co}L \in \text{co}C_2$. □

Note: co$C$ is very different from languages not in C.

Important Complement Classes

- co**P**
  For deterministic Turing Machines, flip accept and reject states. As a result **P**= co**P**.
- co**PSPACE**
  As with **P**and co**P**, **PSPACE**= co**PSPACE**
- **NPSPACE**
  By Savitch's theorem **NPSPACE**= **PSPACE**, so co**NPSPACE**= co**PSPACE**= **PSPACE**
- What about co**NP**?

Closure under Complements of Nondeterministic Space Classes

co$\mathbf{SPACE}(s(n)) \subseteq$ co$\mathbf{NSPACE}(s(n))$ follows from previous theorem. Combining with Savitch's theorem, we have

$$\mathbf{NSPACE}(s(n)) \subseteq \mathbf{SPACE}(s(n)^2) \subseteq \text{co}\mathbf{NSPACE}(s(n)^2)$$

If the space to recognize a set of languages is at least logarithmic, a stronger result is known:

Theorem (Immerman-Scelepscenyi)

If $r(n) = \Omega(\log n)$ is proper,

$$\mathbf{NSPACE}(r(n)) = co\mathbf{NSPACE}(r(n))$$

co**NP**

Theorem

Let L be an **NP***-complete language. co$L$ is co***NP***-complete.*

Proof

By definition, if $L \subseteq \Sigma_1^*$ is in **NP**, co$L \subseteq \Sigma_2^*$ is in co**NP**. Any language L' in **NP** can be reduced to L using some polynomial time reduction, $f(x)$.

f also reduces $\bar{L}'$ to $\bar{L}$ in PTIME. By assumption the sets of instances of L' and L, denoted $WF_{L'}$ and WF_L, respectively, are PTIME recognizable. It follows that there exists a PTIME computable function $g : \Sigma_1^* \mapsto \Sigma_2^*$ such that $x \in$ co$L' \Leftrightarrow g(x) \in$ co$L$ obtained by rejecting strings not in $WF_{L'}$ and applying f to those in $WF_{L'}$.

It is not known whether **NP**= co**NP**. Notice that if **NP** $\neq$ co**NP**, then **P** $\neq$ **NP**. This provides another possible test to determine whether **NP** and **P** are the same or different.

Languages in co**NP**

- The language **BF** is the set of Boolean formulas $\{b(\mathbf{x})\}$ such that $\exists \mathbf{u}\ b(\mathbf{u})$. (There exists $\mathbf{u}$ such that $b(\mathbf{u})$ is True.)
- co**BF** is thus $\neg\exists \mathbf{u}\ b(\mathbf{u}) = \forall \mathbf{u}\ \bar{b}(\mathbf{u})$, but $\bar{b}(\mathbf{u})$ is a boolean formula $b'(\mathbf{u})$.
- Since SAT is **NP**-complete, so is **BF**. co**BF** is co**NP**-complete.
- co**BF** or TAUTOLOGY can be described as the set of formulas such that $\forall \mathbf{u}\ b(\mathbf{u})$.

Polynomial Time Hierarchy

- A language is in **NP**(co**NP**) if and only if it can be reduced in polynomial time to a statement of the form $\exists \mathbf{x}\ b(\mathbf{x})$ $(\forall \mathbf{x}\ b(\mathbf{x}))$
- What about additional levels of alternation?
  - $\forall \mathbf{x}_1\ \exists \mathbf{x}_2\ b(\mathbf{x}_1, \mathbf{x}_2)$
  - $\exists \mathbf{x}_1\ \forall \mathbf{x}_2\ b(\mathbf{x}_1, \mathbf{x}_2)$
- The sets of languages reducible to statements of this form are denoted Π_2^p and Σ_2^p respectively.
- More generally, we can consider any constant number of alternations and denote these sets of languages Π_i^p and Σ_i^p. Here Π (Σ) signals that the outermost operator is universal (existential) quantification. The subscript indicates the number of levels of quantification.

The Polynomial Time Hierarchy

Definition

The Polynomial Hierarchy (**PH**) is defined as

$$\mathbf{PH} = \bigcup_i \Sigma_i^p$$

- Just as it is believed that $\mathbf{P} \neq \mathbf{NP}$ and co$\mathbf{NP} \neq \mathbf{NP}$, it is conjectured that all levels of **PH** are distinct.
- As with SAT and TAUTOLOGY, notice that $\Pi_i^p = \mathrm{co}\Sigma_i^p$.
- If for i, $\Pi_i^p = \Sigma_i^p$, $\mathbf{PH} = \Sigma_i^p$, meaning **PH** collapses at ith level. E.g. if $\forall_{\mathbf{u}_3} \exists_{\mathbf{u}_2} \forall_{\mathbf{u}_1} b(\mathbf{u}_3, \mathbf{u}_2, \mathbf{u}_1) = \exists_{\mathbf{u}_3} \forall_{\mathbf{u}_2} \exists_{\mathbf{u}_1} b(\mathbf{u}_3, \mathbf{u}_2, \mathbf{u}_1)$ then

$$\begin{aligned} \exists_{\mathbf{u}_4} \forall_{\mathbf{u}_3} \exists_{\mathbf{u}_2} \forall_{\mathbf{u}_1} b(\mathbf{u}_3, \mathbf{u}_2, \mathbf{u}_1) &= \exists_{\mathbf{u}_4} \exists_{\mathbf{u}_3} \forall_{\mathbf{u}_2} \exists_{\mathbf{u}_1} b(\mathbf{u}_3, \mathbf{u}_2, \mathbf{u}_1) \\ & \exists_{\mathbf{u}_4, \mathbf{u}_3} \forall_{\mathbf{u}_2} \exists_{\mathbf{u}_1} b(\mathbf{u}_3, \mathbf{u}_2, \mathbf{u}_1) \end{aligned}$$

- If $\mathbf{P} = \mathbf{NP}$, $\Pi_1^p = \Sigma_1^p$ and $\mathbf{PH} = \mathbf{P}$.

PH and PSPACE

- It is not hard to see that **PH** $\in$ **PSPACE**.
- A language L is **PH-complete** if $L \in$ **PH** and all languages in **PH** are PTIME reducible to L.

Theorem

If there is a language that is **PH***-complete, the polynomial hierarchy collapses, that is, for some* i, **PH** $\subseteq \Sigma_i^p$.

Proof.

To see why, since **PH** $= \bigcup_i \Sigma_i^p$ there is some i such that $L \in \Sigma_i^p$. Since L is **PH**-complete, we can reduce every language in **PH** to it and to Σ_i^p. Thus, **PH** $\subseteq \Sigma_i^p$. □

More Alternations

- An instance of TQBF is a quantified boolean formula with an unbounded number of alternations. In otherwords, each variable can be quantified separately.
- Any language in **PH** can be reduced to TQBF.
- TQBF $\in$ **PSPACE**.
- Later, we show TQBF is **PSPACE**-complete, that is, all languages in **PSPACE**can be reduced to TQBF in polynomial time.