CSCI 1590
Intro to Computational Complexity
Formula Size

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March 5, 2008

Summary

1. Review
2. Application of Neciporuk's Lower Bound
3. Krapchenko's Formula Size Lower Bound

Review of Results Concerning Circuit Size

Lemma

For $n \geq 3$, $Q_{2,3}^{(n)}$ *contains* $f_{\text{mod } 3,c}^{(n)}$ *where*

$$f_{\text{mod } 3,c}^{(n)}(x_1, x_2, \ldots, x_n) = ((y + c) \text{ mod } 3) \text{ mod } 2)$$

for $c \in \{0, 1, 2\}$ *where* $y = x_1 + x_2 + \ldots + x_n$.

Theorem

Over the basis of all Boolean functions on 2 inputs, $f \in Q_{2,3}^{(n)}$ *for* $n \geq 3$ *has* $C(f) \geq 2n - 3$.

Review of Results Concerning Formula Size

Definition

Given $f : \mathcal{B}^n \mapsto \mathcal{B}$, partition its n variables $\mathbf{X}$ into p disjoint sets $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_p$.

Let $r_j(f)$ be the number of different subfunctions of f over $\mathbf{X}_j$ when the variables in $\mathbf{X} - \mathbf{X}_j$ range over all values.

Theorem

Let Ω be a complete basis of fan-in d and let $c_\Omega = 1/(d+2)$. Then, for every $f : \mathcal{B}^n \mapsto \mathcal{B}$ its formula size satisfies

$$L_\Omega(f) \geq c_\Omega \sum_{i=1}^{p} \log_2 r_i(f)$$

Neciporuk's Lower Bound to Formula Size

Indirect Storage Access Function

$f_{ISA}(k,l) : \mathcal{B}^n \mapsto \mathcal{B}$ has $n = k + l2^k + 2^l$, $\mathbf{a} \in B^k$, $\mathbf{x}_j \in B^l$, $\mathbf{y} \in B^{2^l}$.

$$f_{ISA}^{(k,l)}(\mathbf{a}, \mathbf{x}_{2^k-1}, \ldots, \mathbf{x}_0, \mathbf{y}) = y_{|\mathbf{x}_{|\mathbf{a}|}|}$$

$$L_\Omega(f_{ISA}^{(k,l)}) = O(\frac{n^2}{\log n})$$

where $n = k + l2^k + 2^l$. (See Problem 9.24 of Savage's book.)

Lemma

$$L_\Omega(f_{ISA}^{(k,l)}) = \Omega(\frac{n^2}{\log n})$$

Lower Bounds to Formula Size

Proof.

Let $K = 2^k$, $L = 2^l$. Create K sets $\mathbf{X}_1, \ldots, \mathbf{X}_K$. Let $\mathbf{X}_j$ be the variables in $\mathbf{x}_j$ and possibly other variables that are fixed. (This cannot increase $r_j(f)$.)

$r_j(f) \geq 2^{2^l}$ because when $|\mathbf{a}| = j$ each of the 2^{2^l} assignments of $\mathbf{y}$ defines a different function of the variables in $\mathbf{X}_j$. It follows that $L_\Omega(f_{ISA}^{(k,l)}) \geq c_\Omega \sum_{i=1}^{p} \log_2 r_i(f) \geq c_\Omega KL$.

Let $k = \lceil \log_2 \frac{L}{l} \rceil$. Then, $L/l \leq K = 2^k < 2L/l$. Since, $l = \log_2 L$, $n = \log_2 K + K \log_2 L + L$ satisfies the following bounds.

$$2L \leq n \leq \lceil \log_2 \frac{L}{l} \rceil + 3L \leq 4L$$

Thus, $n = \Theta(L)$ and $L_\Omega(f_{ISA}^{(k,l)}) \geq c_\Omega L^2 / \log_2 L = \Omega(n^2 / \log_2 n)$, the desired result. ☐

Krapchenko's Formula Size Lower Bound

- Krapchenko's lower bound to formula size applies to the bases:
  - a) standard, b) $\{\text{AND}, \text{NOT}\}$, and c) $\{\text{OR}, \text{NOT}\}$.
- It provides slightly stronger results than Neciporuk's.
- We use Krapchenko's method to show that parity on n inputs satisfies $L(f_{\oplus}^{(n)}) \geq \Omega(n^2)$.

Krapchenko's Formula Size Lower Bound

We now give an upper bound on formula size for $f_{\oplus}^{(n)}$. $L(f_{\oplus}^{(2)}) \leq 4$ because the following works on two inputs:

$$f_{\oplus}^{(2)}(x_1, x_2) = (x_1 \wedge \bar{x}_2) \vee (\bar{x}_1 \wedge x_2)$$

We compute $f_{\oplus}^{(n)}$ on n variables by separately computing it on each half of the variables and then combining the results with the above formula.

If $n = 2^k$, let $P(k) = L(f_{\oplus}^{(n)})$. It follows from above formula that $P(k) = 4 * P(k-1) = 4^k = n^2$. The upper bound will agree with the lower bound for this function.

Krapchenko's Formula Size Lower Bound

Definition

Let A and B be disjoint sets of B^n. The **neighborhood** of A and B, $N(A, B)$, is the set of pairs (x, y) of n-tuples, $x \in A$ and $y \in B$, that differ in exactly one position.

For $f : \mathcal{B}^n \mapsto \mathcal{B}$ let $f^{-1}(0)$ and $f^{-1}(1)$ be the sets of n-tuples that cause f to assume values 0 and 1, respectively.

Theorem

For $f : \mathcal{B}^n \mapsto \mathcal{B}$ and any $A \subseteq f^{-1}(0)$ and any $B \subseteq f^{-1}(1)$ the following holds over the standard basis:

$$L(f) \geq \frac{|N(A, B)|^2}{|A||B|}$$

Krapchenko's Formula Size Lower Bound

Proof

For 2-input bases the number of leaves in the optimal formula is one more than the number of two-input gates. We assume w.l.o.g. that the gates used consist of AND and NOT.

The proof is by induction. Base case: $n = 1$. Here f is either a constant ($|N(A,B)| = 0$ and $L(f) = 0$) or it is x or $\bar{x}$. If $f(x) = x$, $f^{-1}(0) = 0$ and $f^{-1}(1) = 1$, and with $A = \{0\}$, $B = \{1\}$ we have $|N(A,B)| = 1$. A similar argument holds if $f(x) = \bar{x}$. Since $|A| = |B| = 1$, the base case holds.

We assume that $L(f) \geq |N(A,B)|^2/|A||B|$ holds for f such that $L(f) \leq L_0 - 1$ and show that it holds for $L(f) = L_0 \geq 2$.

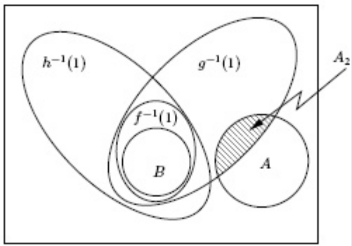
W.l.o.g assume that the output gate is AND and computes $f = g \wedge h$. We assume the formula for f is optimal. This must also be true for g and h.

Krapchenko's Formula Size Lower Bound

Proof (cont.)

Let $A \subseteq f^{-1}(0)$ and $B \subseteq f^{-1}(1)$. That is, $f(x) = 0$ for $x \in A$ and $f(x) = 1$ for $x \in B$.

It follows that $g(x) = h(x) = 1$ when $x \in B$. That is, $f^{-1}(1) \subseteq g^{-1}(1)$ and $f^{-1}(1) \subseteq h^{-1}(1)$. Thus, $B \subseteq g^{-1}(1)$ and $B \subseteq h^{-1}(1)$. Let $B_1 = B_2 = B$. Let $A_1 = A \cap g^{-1}(0)$ and $A_2 = A - A_1$.



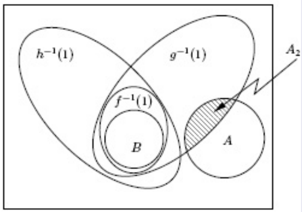
Krapchenko's Formula Size Lower Bound

Proof (cont.)

If $x \in A$, $f(x) = 0$. If x is also in A_2, $g(x) = 1$. This implies $h(x) = 0$ or $A_2 \subseteq h^{-1}(0)$ which implies $N(A_1, B_1)$ and $N(A_2, B_2)$ are disjoint (recall that $B_1 = B_2 = B$) and $N(A, B) = N(A_1, B_1) + N(A_2, B_2)$.
Because g and h are independent, by induction

$$L(g) = |N(A_1, B_1)|^2 / |A_1||B_1|$$

$$L(h) = |N(A_2, B_2)|^2 / |A_2||B_2|$$



Krapchenko's Formula Size Lower Bound

Proof (cont.)

Also, $L(f) = L(g) + L(h)$. In the above we have $|B|$ as common denominator.

The remaining expression is of the form

$$S = (n_1^2/c_1) + (n_2^2/c_2)$$

where $n_1 + n_2 = N(A, B)$, $c_1 = |A_1|$, and $c_2 = |A_2|$. It is straightforward to show that

$$S \geq (n_1 + n_2)^2/(c_1 + c_2)$$

from which the result follows.