Review

Converted integration over areas into integration over space of all paths.

- Wanted to compute, for a particular pixel,

$$m = \int_{M \times M} W_e(x_1 \to x_2) L(x_1 \to x_2) G(x_1 \leftrightarrow x_2) dA(x_1) dA(x_2)$$

where W_e measures response of sensor to light travelling from x_1 to x_2; if the ray $x_1 x_2$ passes nowhere near the pixel, this is zero.

- Instead, rewrite this as an integral over

$$\Omega = \Omega_2 \cup \Omega_3 \cup \ldots$$

where Ω_2 = all two-step paths, i.e., paths of the form $x_1 x_2$, where each x_i is a point of some surface in the scene, and similarly for Ω_3, etc.

What to integrate

- On this large "path space" we needed to integrate a function defined on each piece by the general formula

$$f(x_1 x_2 \dots x_n) = L(x_1 \to x_2)$$

$$\prod_{i=1}^{n-2} G(x_i \leftrightarrow x_{i+1}) f(x_i \to x_{i+1} \to x_{i+2})$$

$$G(x_{n-1} \leftrightarrow x_n) W_e(x_{n-1} \to x_n)$$

- Special case:

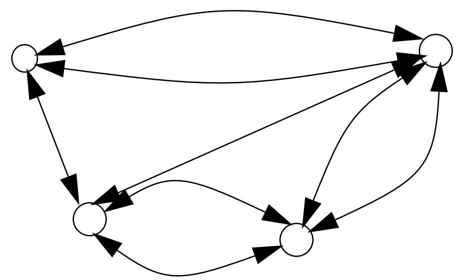
$$f(x_1 x_2) = L(x_1 \to x_2) G(x_1 \leftrightarrow x_2) W_e(x_1 \to x_2)$$

How to integrate

- Jump around in the space Ω via metropolis.
- Desired probability density for path x is just $f(x)$

Metropolis worked via detailed balance

- General idea was to define $a(y|x)$ so that the "flow" along the edge from x to y was the same as the flow in the reverse direction (this is a discretized explanation)
- Suppose you had multiple paths from x to y



- As long as there is "detailed balance" *on each edge* you get detailed balance overall.

Large-scale structure of code

```
N times {

    y = Mutate (x)
    a = AcceptProb(y | x)
    if Random() < a
        x = y
    RecordSample(image, x)
}
```

Large-scale structure of code

```
N times {
    choose mutation strategy i
    y = Mutate(i, x)
    a = AcceptProb(i, y | x)
    if Random() < a
        x = y
    RecordSample(image, x)
}
```

- As long as for each strategy i, we have

$$f(x)T_i(y|x)a_i(y|x) = f(y)T_i(x|y)a_i(x|y)$$

everything works out fine.

Last time: bidirectional mutation

PART A: Mutating

STEP 1:

- given a current path $x = x_0 x_1 \ldots x_k$, choose a segment to replace. Pick two numbers s and t, and delete $x_s \ldots x_t$.
- This leaves behind $x_0 \ldots x_s$ and $x_t \ldots x_k$
- Possible values for s and t are ???

STEP 2:

- Choose length l of path to be inserted

STEP 3:

- Choose a value s' between 0 and l, and let $t' = l - s'$.
- Add s' vertices to the start segment and t' verts to the end segment.

Adding Vertices

- Given a path $x_0 x_1 \ldots x_p$ add a vertex after x_p by...
- 1. Selecting a direction from the BSDF at x_p
- 2. Tracing a ray along that direction to the first hit
- 3. Calling the hit point x_{p+1}

- Special cases
  - if $p = -1$, i.e., the subpath is empty. Choose a point on any light surface in the world
  - dual situation: if you're filling in

$$x_p x_{p+1} \ldots x_k$$

    to get x_{p-1} (i.e. the lens subpath), there's a problem if $p = k + 1$; choose any point on the lens.

Adding Vertices (2)

- After adding s' verts to the start and t' verts to the end, we've got a path...

$$x_0 \ldots x_s \ldots x_{s+s'} \qquad x_{t-t'} \ldots x_k$$

- Check whether the first point of the last part is visible from last point of first part.
- If not, then $f = 0$ and path will be rejected.
- If so, it's a potential mutation.

Probabilities

Choosing a segment to delete

- From the paper:

$$p_d[s, t] = p_{d,1}[t - s]p_{d,2}[s, t]$$

- p_d is the probability of deleting the part between x_s and x_t.
- $p_{d,1}$ selects how large a path to delete

$$p_{d,1}(e) = \begin{cases} \frac{1}{4} & e = 1 \\ \frac{1}{2} & e = 2 \\ 2^{-e} & e > 2 \end{cases}$$

- $p_{d,2}$ determines which one (of that length) to delete
- Note that this favors quite modest mutations

Probabilities (2)

Choosing a segment to add

- We've chosen a segment to delete, of length l_d.
- We'll add a segment of some length e according to

$$p_a(e) = \begin{cases} \frac{1}{2} & e = l_d \\ 0.15 & e = l_d - 1 \\ 0.15 & e = l_d + 1 \\ remainder & others \end{cases}$$

Intermediate Summary

- We know how to get from an old path to a new one, except that we don't *which* segment of the old path to delete
- We also need to sample from the BSDF.

Digression: Sampling from the BSDF

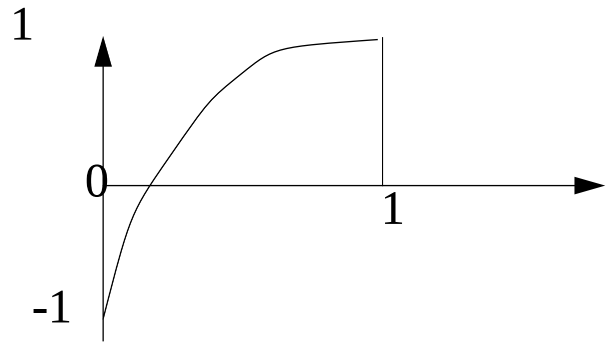
- You've got a function $f_s(x_1, x_2, x_3)$, and you're given x_1 and x_2;
- Therefore have a function of just x_3.
- Need to pick x_3 with distribution given by f_s.
- How?

Sampling wrt BSDF

- Too hard in general
- If reflection is specular, you just choose reflected ray
- If reflection is NOT specular, approximate with Phong model, or something near it.
- Why is this good enough?
- Because you'll be incorporating your sampling method into $T(y|x)$; the actual VALUE of the BSDF will appear in $f(y)$, hence in $a(y|x)$

Phong-like sampling

- Given "target" ray, find a ray likely to be near target and compute probability that you selected that one ray.
- WLOG, assume target is z axis $(0, 0, 1)$.
- Method:
  - pick random number r in $[0, 1]$.
  - Write $z = f(r)$ where f is some function like this



  - Pick θ randomly in $[0, 2\pi]$.
  - Generate point

$$(\cos\theta\sqrt{1-z^2}, \sin\theta\sqrt{1-z^2}, z)$$

  - Probability of doing so is $\dfrac{1}{2\pi f'(r)}$

Pending Problem: which segment to delete

Back to Mutations

Part A: mutating -- done except for delete-seg picking
Part B: accepting

- Need to compute $T(y|x)$ and $f(x)$ and $f(y)$.

AcceptProb(y | x), cleverly written

```
acceptProb(y,x)
{
    r1 = R(y, x);
    r2 = R(x, y);
    return r1/r2;
}

R(x,y)
{
    return (fd(y,x))/T(y,x)
}

fd(y,x) // part of f(y) different from f(x)
{
    Let fc = common part of f(x) and f(y)
    return f(y) / fc
}
```

Computing fd(y,x)

- $fd(y, x)$ is the "physics stuff" for y that isn't in that for x
- For a path $x_1 x_2 \ldots x_n$ this "physics stuff" looks like

$$L_e(x_0, x_1)\left(\prod_{i=0}^{n-2} G(x_i, x_{i+1}) f_s(x_i, x_{i+1}, x_{i+2})\right) G(x_{n-1}, x_n) W(x_{n-1}, x_n)$$

- Call this $f(x)$.
- If

$$x = x_0 x_1, \ldots, x_k, u_1, u_2, \ldots, u_p, x_{k+p+1}, \ldots, x_n$$

$$y = x_0 x_1, \ldots, x_k, v_1, v_2, \ldots, v_s, x_{k+p+1}, \ldots, x_n$$

- Then $f(y)$ and $f(x)$ have lots of identical terms. Let $u_0 = x_k$ and $u_{p+1} = x_{k+p+1}$, and similarly for v.
- Then

$$fd = \prod_{i=0}^{s-1} G(v_i, v_{i+1}) f_s(v_i, v_{i+1}, v_{i+2})$$

(except in special case where $k = -1$ or $k + p = n$).

Computing f(y,x)

```
f(path y, path x)
{
    find differing segments of x and y; suppose that they
    start differing at x[k], start agreeing again at x[k+p+1]

    if (k == -1) { // special case}
    if (k + p + 1 == max[x]) { // special case}

    Let u[0] be the last point at which they agree

    call the “differing subsequences”
        u[0] ... u[p+1],
        v[0] .. v[s+1]

    a1 = gf(u);
    a2 = gf(v);
    return a2/a1;
}

gf(path u)
{
    double p = 1.0;
    for (i = 0; i < max[u]-1; i++ ) {
        p *= visible(u[i], u[i+1]);
        p *= u[i+1].bsdf(u[i], u[i+1], u[i+2]);
        if (p == 0.0) break;
    }
    return p;
}
```

Recall our goal

- Need to compute $a(y\,|\,x) = \dfrac{R(y\,|\,x)}{R(x\,|\,y)}$
- Recall $R(y\,|\,x) = \dfrac{f(y)}{T(y\,|\,x)}$, where f is the integrand, and T is the probability of jumping from x to y.
- We've dealt with $f(y)$…but what about $T(y|x)$?
- Important note: we got from x to y through a certain route -- we chose s, t, etc., and hit upon y. But perhaps if we'd chosen $s-1$, $t+1$, we could have hit upon y differently.
- $T(y\,|\,x)$ must take all possible routes into account!

An example (more to follow)

- Starting x:

$$x = x_0 x_1 x_2 x_3$$

- "Deletion" chooses $s = 1$ $t = 2$; "addition" chooses $s' = 1$, $t' = 0$.
  - Deletion probability: 1/4
  - Addition probability: 1/2
- After deletion $x_0 x_1 \qquad x_2 x_3$.
- After addition: $x_0 x_1 z_0 x_2 x_3$.
- Let $p_s(x_0, x_1, z_0)$ be the probability density of sampling the $x_1 z_0$ direction, wrt projected solid angle.
- Then prob. of sampling z_0 is $p_s(x_0, x_1, z_0) G(x_0, x_1)$.
- $R(y \mid x) = \dfrac{f(y)}{T(y \mid x)}$; numerator is

$$f(y) = f_s(x_0, x_1, z_0) G(x_1, z_0) f_s(x_1, z_0, x_2) G(z_0, x_2) f_s(z_0 x_2 x_3)$$

(factors shared by $R(x, y)$ and $R(y, x)$ dropped!)

Denominator

Finding $T(y \mid x)$

- How did we get to y from x? We had to do the right deletion, the right insertion, and select the right point (z_0).

$$T(y \mid x) = p_d[1, 2]U$$

$$U = p_a[1, 0]p_s(x_0, x_1, z_0)G(x_1, z_0) + p_a[0, 1]p_s(x_3, x_2, z_0)G(x_2, z_0)$$

- Similarly,

$$R(x \mid y) = \frac{f_s(x_0, x_1, x_2)G(x_1, x_2)f_s(x_1, x_2, x_3)}{p_d[1, 3]p_a[0, 0]}$$

Precomputation

- Because fGf terms keep appearing, as do $\frac{f_s}{p_s}$ terms, one can precompute these to simplify computations

$$C(x_0, x_1, x_2, x_3) = f_s(x_0, x_1, x_2)G(x_1, x_2)f_s(x_1, x_2, x_3)$$

$$S(x_0, x_1, x_2) = \frac{f_s(x_0, x_1, x_2)}{p_s(x_0, x_1, x_2)}$$

- Note that for specular reflections, $\frac{f_s}{p_s} = 1$, whereas prob. densities don't quite make sense.

Pending

- In p_d, there were two terms; we have yet to describe the second one
- There are two other mutation strategies to be considered

Example I fleshed out... the missing pd term

- Start with $x = x_0 x_1 x_2 x_3$
- "Deletion" chooses $s = 1$ $t = 2$; "addition" chooses $s' = 1$, $t' = 0$.
- After deletion: $x_0 x_1 \qquad x_2 x_3$.
- After addition: $x_0 x_1 z_0 x_2 x_3$.
- Let $p_s(x_0, x_1, z_0)$ be the probability density of sampling the $x_1 z_0$ direction, wrt projected solid angle.
- Then prob. of sampling z_0 is $p_s(x_0, x_1, z_0) G(x_0, x_1)$.
- $R(y \mid x) = \dfrac{f(y)}{T(y \mid x)}$; numerator is

$$f(y) = f_s(x_0, x_1, z_0) G(x_1, z_0) f_s(x_1, z_0, x_2) G(z_0, x_2) f_s(z_0 x_2 x_3)$$

- Factors shared by $R(x, y)$ and $R(y, x)$ dropped. They are

$$L_e(x_0 \to x_1), G(x_2, x_3), W_e(x_2, x_3)$$

Denominator

Finding $T(y \mid x)$

- How did we get to y from x? We had to do the right deletion, the right insertion, and select the right point (z_0).

$$T(y \mid x) = p_d[1, 2]U$$

$$U = p_a[1, 0]p_s(x_0, x_1, z_0)G(x_1, z_0) + p_a[0, 1]p_s(x_3, x_2, z_0)G(x_2, z_0)$$

- Similarly, $R(x \mid y) = \dfrac{f_s(x_0, x_1, x_2)G(x_1, x_2)f_s(x_1, x_2, x_3)}{p_d[1, 3]p_a[0, 0]}$

- fGf terms keep appearing; so do $\dfrac{f_s}{p_s}$ terms; precompute to simplify computations

$$C(x_0, x_1, x_2, x_3) = f_s(x_0, x_1, x_2)G(x_1, x_2)f_s(x_1, x_2, x_3)$$

$$S(x_0, x_1, x_2) = \frac{f_s(x_0, x_1, x_2)}{p_s(x_0, x_1, x_2)}$$

Summary

$$R(y\,|\,x) = \frac{f_s(x_0, x_1, z_0)G(x_1, z_0)f_s(x_1, z_0, x_2)G(z_0, x_2)f_s(z_0 x_2 x_3)}{p^x{}_d[1, 2](p^x{}_a[1, 0]p_s(x_0, x_1, z_0)G(x_1, z_0) + p^x{}_a[0, 1]p_s(x_3, x_2, z_0)G(x_2, z_0))}$$

$$R(x\,|\,y) = \frac{f_s(x_0, x_1, x_2)G(x_1, x_2)f_s(x_1, x_2, x_3)}{p^y{}_d[1, 3]p^y{}_a[0, 0]}$$

- Using

$$C(x_0, x_1, x_2, x_3) = f_s(x_0, x_1, x_2)G(x_1, x_2)f_s(x_1, x_2, x_3)$$

$$S(x_0, x_1, x_2) = \frac{f_s(x_0, x_1, x_2)}{p_s(x_0, x_1, x_2)}$$

rewrite

$$R(y\,|\,x) = \frac{C(x_0, x_1, z_0, x_2)C(x_1, z_0, x_2, x_3)/f_s(x_1, z_0, x_2)}{p^x{}_d[1, 2](p^x{}_a[1, 0]p_s(x_0, x_1, z_0)G(x_1, z_0) + p^x{}_a[0, 1]p_s(x_3, x_2, z_0)G(x_2, z_0))}$$

Continued

$$S(x_0, x_1, x_2) = \frac{f_s(x_0, x_1, x_2)}{p_s(x_0, x_1, x_2)}$$

$$R(y \mid x) = \frac{C(x_0, x_1, z_0, x_2)C(x_1, z_0, x_2, x_3) / f_s(x_1, z_0, x_2)}{p^x{}_d[1, 2](p^x{}_a[1, 0]p_s(x_0, x_1, z_0)G(x_1, z_0) + p^x{}_a[0, 1]p_s(x_3, x_2, z_0)G(x_2, z_0))}$$

Continued

$$S(x_0, x_1, x_2) = \frac{f_s(x_0, x_1, x_2)}{p_s(x_0, x_1, x_2)}$$

$$R(y \mid x) = \frac{C(x_0, x_1, z_0, x_2)C(x_1, z_0, x_2, x_3) / f_s(x_1, z_0, x_2)}{p^x{}_d[1, 2](p^x{}_a[1, 0]p_s(x_0, x_1, z_0)G(x_1, z_0) + p^x{}_a[0, 1]p_s(x_3, x_2, z_0)G(x_2, z_0))}$$

$$= \frac{C(x_0, x_1, z_0, x_2)C(x_1, z_0, x_2, x_3) / f_s(x_1, z_0, x_2)}{p^x{}_d[1, 2]\left(p^x{}_a[1, 0]\frac{f_s(x_0, x_1, z_0)G(x_1, z_0)}{S(x_0, x_1, z_0)} + p^x{}_a[0, 1]\frac{f_s(x_3, x_2, z_0)G(x_2, z_0)}{S(x_3, x_2, z_0)}\right)}$$

Continued

$$S(x_0, x_1, x_2) = \frac{f_s(x_0, x_1, x_2)}{p_s(x_0, x_1, x_2)}$$

$$R(y \mid x) = \frac{C(x_0, x_1, z_0, x_2)C(x_1, z_0, x_2, x_3) / f_s(x_1, z_0, x_2)}{p^x{}_d[1, 2](p^x{}_a[1, 0]p_s(x_0, x_1, z_0)G(x_1, z_0) + p^x{}_a[0, 1]p_s(x_3, x_2, z_0)G(x_2, z_0))}$$

$$= \frac{C(x_0, x_1, z_0, x_2)C(x_1, z_0, x_2, x_3) / f_s(x_1, z_0, x_2)}{p^x{}_d[1, 2]\left(p^x{}_a[1, 0]\frac{f_s(x_0, x_1, z_0)G(x_1, z_0)}{S(x_0, x_1, z_0)} + p^x{}_a[0, 1]\frac{f_s(x_3, x_2, z_0)G(x_2, z_0)}{S(x_3, x_2, z_0)}\right)}$$

$$= \frac{C(x_0, x_1, z_0, x_2)C(x_1, z_0, x_2, x_3) / f_s(x_1, z_0, x_2)}{K\left(A\frac{f_s(x_0, x_1, z_0)G(x_1, z_0)f_s(x_1, z_0, x_2)}{S(x_0, x_1, z_0)f_s(x_1, z_0, x_2)} + B\frac{f_s(x_3, x_2, z_0)G(x_2, z_0)f_s(x_1, z_0, x_2)}{S(x_3, x_2, z_0)f_s(x_1, z_0, x_2)}\right)}$$

Continued

$$S(x_0, x_1, x_2) = \frac{f_s(x_0, x_1, x_2)}{p_s(x_0, x_1, x_2)}$$

$$R(y \mid x) = \frac{C(x_0, x_1, z_0, x_2) C(x_1, z_0, x_2, x_3) / f_s(x_1, z_0, x_2)}{p^x{}_d[1, 2](p^x{}_a[1, 0] p_s(x_0, x_1, z_0) G(x_1, z_0) + p^x{}_a[0, 1] p_s(x_3, x_2, z_0) G(x_2, z_0))}$$

$$= \frac{C(x_0, x_1, z_0, x_2) C(x_1, z_0, x_2, x_3) / f_s(x_1, z_0, x_2)}{p^x{}_d[1, 2]\left(p^x{}_a[1, 0] \frac{f_s(x_0, x_1, z_0) G(x_1, z_0)}{S(x_0, x_1, z_0)} + p^x{}_a[0, 1] \frac{f_s(x_3, x_2, z_0) G(x_2, z_0)}{S(x_3, x_2, z_0)}\right)}$$

$$= \frac{C(x_0, x_1, z_0, x_2) C(x_1, z_0, x_2, x_3) / f_s(x_1, z_0, x_2)}{K\left(A \frac{f_s(x_0, x_1, z_0) G(x_1, z_0) f_s(x_1, z_0, x_2)}{S(x_0, x_1, z_0) f_s(x_1, z_0, x_2)} + B \frac{f_s(x_3, x_2, z_0) G(x_2, z_0) f_s(x_1, z_0, x_2)}{S(x_3, x_2, z_0) f_s(x_1, z_0, x_2)}\right)}$$

$$= \frac{C(x_0, x_1, z_0, x_2) C(x_1, z_0, x_2, x_3) / f_s(x_1, z_0, x_2)}{K\left(A \frac{C(x_0, x_1, z_0, x_2)}{S(x_0, x_1, z_0)} + B \frac{C(x_3, x_2, z_0, x_1)}{S(x_3, x_2, z_0)}\right) \frac{1}{f_s(x_1, z_0, x_2)}}$$

Continued

$$S(x_0, x_1, x_2) = \frac{f_s(x_0, x_1, x_2)}{p_s(x_0, x_1, x_2)}$$

$$R(y \mid x) = \frac{C(x_0, x_1, z_0, x_2)C(x_1, z_0, x_2, x_3) / f_s(x_1, z_0, x_2)}{p^x{}_d[1, 2](p^x{}_a[1, 0]p_s(x_0, x_1, z_0)G(x_1, z_0) + p^x{}_a[0, 1]p_s(x_3, x_2, z_0)G(x_2, z_0))}$$

$$= \frac{C(x_0, x_1, z_0, x_2)C(x_1, z_0, x_2, x_3) / f_s(x_1, z_0, x_2)}{p^x{}_d[1, 2]\left(p^x{}_a[1, 0]\frac{f_s(x_0, x_1, z_0)G(x_1, z_0)}{S(x_0, x_1, z_0)} + p^x{}_a[0, 1]\frac{f_s(x_3, x_2, z_0)G(x_2, z_0)}{S(x_3, x_2, z_0)}\right)}$$

$$= \frac{C(x_0, x_1, z_0, x_2)C(x_1, z_0, x_2, x_3) / f_s(x_1, z_0, x_2)}{K\left(A\frac{f_s(x_0, x_1, z_0)G(x_1, z_0)f_s(x_1, z_0, x_2)}{S(x_0, x_1, z_0)f_s(x_1, z_0, x_2)} + B\frac{f_s(x_3, x_2, z_0)G(x_2, z_0)f_s(x_1, z_0, x_2)}{S(x_3, x_2, z_0)f_s(x_1, z_0, x_2)}\right)}$$

$$= \frac{C(x_0, x_1, z_0, x_2)C(x_1, z_0, x_2, x_3) / f_s(x_1, z_0, x_2)}{K\left(A\frac{C(x_0, x_1, z_0, x_2)}{S(x_0, x_1, z_0)} + B\frac{C(x_3, x_2, z_0, x_1)}{S(x_3, x_2, z_0)}\right)\frac{1}{f_s(x_1, z_0, x_2)}}$$

$$= \frac{C(x_0, x_1, z_0, x_2)C(x_1, z_0, x_2, x_3)}{K\left(A\frac{C(x_0, x_1, z_0, x_2)}{S(x_0, x_1, z_0)} + B\frac{C(x_3, x_2, z_0, x_1)}{S(x_3, x_2, z_0)}\right)}$$

Deletion Probability

- Based on acceptance function $a(y\,|\,x)$ -- want this to be high.
- Recall

$$a(y\,|\,x) \;=\; \frac{R(y\,|\,x)}{R(x\,|\,y)} \qquad R(y\,|\,x) \;=\; \frac{f(y)}{T(y\,|\,x)}$$

- Note that $a(y\,|\,x)$ is proportional to $\frac{1}{R(x\,|\,y)}$.
- What is $R(x\,|\,y)$?

$$R(x\,|\,y) \;=\; \frac{f(x)}{T(x\,|\,y)} \;=\; \frac{\text{geometry stuff}}{\text{transition stuff}}$$

- The "transition stuff" is from y to x; but y isn't known...give up
- The geometry stuff is fine -- depends only on x.
- Let $p_{d,\,2}$ be computable part of $\frac{1}{R(x\,|\,y)}$!
- In example: $R(x\,|\,y) \;=\; \frac{f_s(x_0, x_1, x_2)G(x_1, x_2)f_s(x_1, x_2, x_3)}{p_d[1, 3]p_a[0, 0]}$, so

$$p_{d,\,2} \;=\; \frac{1}{C(x_0, x_1, x_2, x_3)}$$

Example 2

- Start with $x = x_0 x_1 x_2 x_3 x_4$
- "Deletion" chooses $s = 1$ $t = 3$; "addition" chooses $s' = 1$, $t' = 1$.
- After deletion: $x_0 x_1 \qquad x_3 x_4$.
- After addition: $x_0 x_1 a b x_3 x_4$.
- $R(y \mid x) = \dfrac{f(y)}{T(y \mid x)}$; numerator is ...