CS229 Homework 2: Quaternions

Assignment Out:	Sept. 25th, 2000
Assignment Due:	Sept. 29nd, 2000 (in class)

1. In Hamilton's extension of imaginary numbers, a quaternion is represented as:

$$q = a + bi + cj + dk \tag{1}$$

and the following rules hold:

$$i^2 = j^2 = k^2 = -1 \tag{2}$$

$$\begin{aligned} ij &= k, & ji &= -k \\ jk &= i, & kj &= -i \\ ki &= j, & ik &= -j \end{aligned}$$

Using straightforward algebra and the definitions of dot and cross product to derive the formula for quaternion multiplication. For q_1 given by $(s_1, \mathbf{v_1})$ and q_2 given by $(s_2, \mathbf{v_2})$:

$$q_1 q_2 = (s_1 s_2 - \mathbf{v_1} \cdot \mathbf{v_2}, s_1 \mathbf{v_2} + s_2 \mathbf{v_1} + \mathbf{v_1} \times \mathbf{v_2}) \tag{3}$$

2. Modify spherical linear interpolation (SLERP) to add "ease-in / ease-out" timing to the motion. In other words, the object's angular velocity should smoothly ramp up from zero and ramp down to zero as it rotates from one orientation to the other. Watt and Watt has a description of SLERP that you can use as a reference. Here is another:

(a) Find the quaternion q_{INC} that rotates from q_A to q_B:

$$q_{INC} = q_B q_A^{-1} \tag{4}$$

(b) Extract the axis $\hat{\mathbf{n}}$ and angle θ of this incremental rotation:

$$\begin{aligned} q_{INC} &= (s, \mathbf{v}) = \left(cos\left(\frac{\theta}{2}\right), \sin\left(\frac{\theta}{2}\right) \hat{\mathbf{n}} \right) \\ \theta &= 2cos^{-1}(s) \\ \hat{\mathbf{n}} &= \frac{\mathbf{v}}{\sin\left(\frac{\theta}{2}\right)} \end{aligned}$$

(c) Construct a new incremental rotation at each time t. For time t between 0 and 1, we would have:

$$q_{INC}(t) = \left(cos\left(\frac{\theta t}{2}\right), \sin\left(\frac{\theta t}{2}\right) \hat{\mathbf{n}} \right) \tag{5}$$

(d) Construct the quaternion describing the object's orientation by incrementally rotating from our original orientation q_A:

$$q(t) = q_{INC}(t) \;\; q_A \tag{6}$$

3. **(Bonus)** One reason to move from Euler angles (or fixed axis angles) to quaternions was to avoid singularities (or gimbal lock). One of the costs of this move was that we picked up a fourth parameter. We can add a fourth parameter to Euler angles by adding a fourth axis of rotation. In fact, we can carefully choose this axis so that the original point of gimbal lock is no longer a singularity (e.g. find the axis about which the body cannot rotate when it is in gimbal lock; make this the fourth axis). Does adding a fourth axis of rotation remove gimbal lock completely, or does the system still contain a singularity? Convince me that you are right. (A diagram showing the singularity – should there still be one – for a particular set of axes would be perfect.)