Derivation of Equation 1 in Papadimitriou's Paper Computing Correlated Equilibria in Multi-Player Games

Victor Naroditskiy

April 3, 2008

The coefficients for the term u_{is}^p come from multiplying CE inequalities by y. Let's look at the inequalities in detail. There is an inequality for each player p and pair of strategies $i, j \in S_p$. The total number of constraints for player p is $(|S_p|^2)$. The term u_{is}^p appears in $|S_p|$ of the constraints with the coefficient x_{is} when player p is recommended to play action i. In this case, the constraints ensure that none of the other $|S_p|$ actions (including action i) results in a higher utility. The term u_{is}^p also appears in $|S_p|$ of the constraints with the coefficient $-x_{js}$ when player p is recommended to play action j, and should not want to play action i instead.

Multiplying x^T by U^T results in $(|S_p|^2)$ constraints described above. We are only interested in $2|S_p|$ of those constraints that contain u_{is}^p. The length of y is the same as the number of the constraints. Let's denote by y_{ji}^p the element of y corresponding to the constraint where player p should prefer action i to action j. Multiplying the constraints by the corresponding terms in y, we get

$$\sum_{j \in S_p} x_{is} y_{ji}^p + \sum_{j \in S_p} -x_{js} y_{ij}^p \tag{1}$$

Using the assumption that $x_{is} = x_{s_1}^1 \cdot \ldots \cdot x_{s_n}^n$, we can write the expression above as

$$\begin{aligned} &\sum_{j \in S_p} (\prod_{q \neq p} x_{s_q}^q) x_i^p y_{ji}^p + \sum_{j \in S_p} -(\prod_{q \neq p} x_{s_q}^q) x_j^p y_{ij}^p = \\ &(\prod_{q \neq p} x_{s_q}^q) \left[x_i^p \sum_{j \in S_p} y_{ji}^p - \sum_{j \in S_p} x_j^p y_{ij}^p \right] \end{aligned} \tag{2}$$

Let's look at the game in Table 1.

	1	2
1	u_{11}^1, u_{11}^2	u_{12}^1, u_{12}^2
2	u_{21}^1, u_{21}^2	u_{22}^1, u_{22}^2
3	u_{31}^1, u_{31}^2	u_{32}^1, u_{32}^2

Table 1: Payoff matrix for the example game

The CE constraints for player 1 are shown below:

$$(u_{11}^1 - u_{11}^1)x_{11} + (u_{12}^1 - u_{12}^1)x_{12} \geq 0 \qquad \text{for strategies (1,1)} \tag{3}$$
$$(u_{11}^1 - u_{21}^1)x_{11} + (u_{12}^1 - u_{22}^1)x_{12} \geq 0 \qquad \text{for strategies (1,2)} \tag{4}$$
$$(u_{11}^1 - u_{31}^1)x_{11} + (u_{12}^1 - u_{32}^1)x_{12} \geq 0 \qquad \text{for strategies (1,3)} \tag{5}$$

$$(u_{21}^1 - u_{11}^1)x_{21} + (u_{22}^1 - u_{12}^1)x_{22} \geq 0 \qquad \text{for strategies (2,1)} \tag{6}$$
$$(u_{21}^1 - u_{21}^1)x_{21} + (u_{22}^1 - u_{22}^1)x_{22} \geq 0 \qquad \text{for strategies (2,2)} \tag{7}$$

$$(u^1_{21} - u^1_{31})x_{21} + (u^1_{22} - u^1_{32})x_{22} \geq 0 \qquad \text{for strategies (2,3)} \qquad (8)$$

$$(u^1_{31} - u^1_{11})x_{31} + (u^1_{32} - u^1_{12})x_{32} \geq 0 \qquad \text{for strategies (3,1)} \qquad (9)$$

$$(u^1_{31} - u^1_{21})x_{31} + (u^1_{32} - u^1_{22})x_{32} \geq 0 \qquad \text{for strategies (3,2)} \qquad (10)$$

$$(u^1_{31} - u^1_{31})x_{31} + (u^1_{32} - u^1_{32})x_{32} \geq 0 \qquad \text{for strategies (3,3)} \qquad (11)$$

A constraint for strategies (i, j) ensures that when player 1 is recommended to play strategy i, he does not prefer to switch to strategy j. The system of inequalities above can also be written as $Ux \geq 0$, where $x^T = (x_{11}, x_{12}, x_{21}, x_{22}, x_{31}, x_{32})$. The first row of U is $(u^1_{11} - u^1_{11}, u^1_{12} - u^1_{12}, 0, 0, 0, 0)$. The following rows are set accordingly. There is one row per constraint.

Let's associate with each constraint (i, j) above the variable y^1_{ji}. For example, Eq 10 is associated with y^1_{23}. Now consider the inner product $x^T U^T y$:

$$\begin{aligned}
&[(u^1_{11} - u^1_{11})x_{11} + (u^1_{12} - u^1_{12})x_{12}]y^1_{11} + \\
&[(u^1_{11} - u^1_{21})x_{11} + (u^1_{12} - u^1_{22})x_{12}]y^1_{21} + \\
&[(u^1_{11} - u^1_{31})x_{11} + (u^1_{12} - u^1_{32})x_{12}]y^1_{31} + \\
\\
&[(u^1_{21} - u^1_{11})x_{21} + (u^1_{22} - u^1_{12})x_{22}]y^1_{12} + \\
&[(u^1_{21} - u^1_{21})x_{21} + (u^1_{22} - u^1_{22})x_{22}]y^1_{22} + \\
&[(u^1_{21} - u^1_{31})x_{21} + (u^1_{22} - u^1_{32})x_{22}]y^1_{32} + \\
\\
&[(u^1_{31} - u^1_{11})x_{31} + (u^1_{32} - u^1_{12})x_{32}]y^1_{13} + \\
&[(u^1_{31} - u^1_{21})x_{31} + (u^1_{32} - u^1_{22})x_{32}]y^1_{23} + \\
&[(u^1_{31} - u^1_{31})x_{31} + (u^1_{32} - u^1_{32})x_{32}]y^1_{33}
\end{aligned}$$

Let's look at the coefficients of u^1_{11} in the expression above. The term appears twice on the first line with the coefficients $x_{11}y^1_{11}$ and $-x_{11}y^1_{11}$. It also appears once on lines 2, 3, 4, and 7.

It is easy to confirm that the coefficients of u^1_{11} we just found are also given by Eq 1. Plugging in $i = 1$, $s = 1$, and $p = 1$ in Eq 1, we get

$$\begin{aligned}
&\sum_{j \in S_1} x_{11} y^1_{j1} + \sum_{j \in S_1} -x_{j1} y^1_{1j} = \\
&\sum_{j \in \{1,2,3\}} x_{11} y^1_{j1} + \sum_{j \in \{1,2,3\}} -x_{j1} y^1_{1j} = \\
&x_{11}(y^1_{11} + y^1_{21} + y^1_{31}) - (x_{11}y_{11} + x_{21}y_{12} + x_{31}y_{13})
\end{aligned}$$

Markov Chain

By definition, a stationary distribution of a Markov chain with states S and transition probabilities P is a probability distribution π such that:

$$\pi_i = \sum_{j \in S} \pi_j P_{ji} \qquad \forall i \in S$$

Using the fact that each row of the transition probability matrix sums to one (you have to transition somewhere with probability 1), we write the equation above as

$$\pi_i \sum_{j \in S} P_{ij} = \sum_{j \in S} \pi_j P_{ji} \qquad \forall i \in S$$

or equivalently as

$$\sum_{j \in S} \pi_i P_{ij} = \sum_{j \in S} \pi_j P_{ji} \qquad \forall i \in S$$

The equation above means that in a stationary distribution the probability of the chain leaving a state is the same as the probability of the chain entering the state.

We are going to focus on the second term of Eq 2:

$$\left[x_i^p \sum_{j \in S_p} y_{ji}^p - \sum_{j \in S_p} x_j^p y_{ij}^p \right]$$

We interpret the vector $(x_i^p)_{i \in S_p}$ as the stationary distribution over the states S_p and the matrix $Y^p = (y_{ij}^p)_{i \in S_p, j \in S_p}$ as the transposed matrix of transition probabilities. (Note that the transition probabilities need to be normalized to add up to one per column of the transposed transition probability matrix.)

The Markov chain is finite and therefore has a stationary distribution. But as we just showed, a stationary distribution satisfies the property that the probability of leaving state i is the same as the probability of entering state i (order of indexes is switched because the transition probability matrix is transposed):

$$x_i^p \sum_{j \in S_p} y_{ji}^p = \sum_{j \in S_p} x_j^p y_{ij}^p$$

Therefore, there exists x^p that makes the second term of Eq 2 zero.