Problem Set 1

Problem 1.7

(From Vazirani Ch. 1)

Lemma 1. *The size of a longest chain in A is less than or equal to the size of an antichain cover of A.*

Proof. Let $S \subseteq A$ be a longest chain $\{s_1, \ldots, s_m\}$, $s_1 \leq \cdots \leq s_m$, and let $\mathcal{C} = \{C_1, \ldots, C_n\}$ be an antichain cover of A. Since $\mathcal{C}$ is a partition of A, every s_i must be an element of exactly one antichain C_i. Every pair $s_i, s_j \in S, i \neq j$ are comparable, so $s_i \in C_i$ implies that $s_j \notin C_i$ (by the definition of an antichain). Thus every s_i is associated with its own antichain C_i, and there must be at least as many antichains in $\mathcal{C}$ as there are s_i in S. □

Lemma 2. *The size of a longest chain is greater than or equal to the size of a smallest antichain cover.*

Proof. By Lemma 1, it suffices to show that there is an antichain cover exactly equal in size to the longest chain (since all other antichain covers would necessarily be larger). We follow Vazirani's construction of such an antichain cover. Let the size of the longest chain be m. For $a \in A$, let $\phi(a)$ denote the size of the longest chain in which a is the smallest element. Consider a family $\mathcal{P}$ of sets $A_i = \{a \in A | \phi(a) = i\}$, for $1 \leq i \leq m$. Note that the size of $\mathcal{P}$ is the size of the longest chain. We can demonstrate that $\mathcal{P}$ is an antichain cover of A by showing that it is a pairwise disjoint cover of A and that each A_i is an antichain. Since $1 \leq \phi(a) \leq m$ for all $a \in A$, we know that $A_{\phi(a)}$ exists and that $a \in A_{\phi(a)}$. So every a is covered by some A_i. Since $\phi(a)$ takes on exactly one value for a particular a, it must be that $A_i \cap A_j$ is empty for all $i \neq j$, so A_i are disjoint. Now we assume that there is a comparable pair $a_1, a_2 \in A_i$ for some i and deduce a contradiction. Without loss of generality, assume that $a_1 \leq a_2$. By our definition of A_i, it must be that $\phi(a_1) = \phi(a_2) = i$. Let L be the longest chain in which a_2 is the smallest element (note that the length of L is i). We can create a chain $L' = L \cup \{a_1\}$ whose smallest element is a_1 and whose length is $i+1$, in contradiction to $\phi(a_1) = i$. Therefore there is no comparable pair $a_1, a_2 \in A_i$ for any i, so all A_i are antichains and $\mathcal{P}$ is an antichain cover of A. □