21.6

As the hint suggests, we map each v to the vector $\left(\frac{d(v,S_i)}{Q^{1/p}} \ : \ i = 1, \ldots, Q\right)$. I'll denote $f(u, v)$ to be the ℓ_p distance between the images of u and v, so that

$$f(u,v) = \left(\frac{\sum_{i=1}^{Q} |d(u,S_i) - d(v,S_i)|^p}{Q}\right)^{1/p}.$$

To continue, we emulate certain parts of the proof of Theorem 21.22. Using the same notation as the book, let's say that $B(u, \rho) = \{s \in V \ : \ d(u, s) \leq \rho\}$ and $B^\circ(u, \rho) = \{s \in V \ : \ d(u, s) < \rho\}$. We can also use the same definition of ρ_t; namely it is the smallest radius ρ such that $|B(u, \rho)| \geq 2^t$ and $|B(v, \rho)| \geq 2^t$, and the same definition of $\hat{t} = \max\{t \ : \ \rho_t < d(u, v)/2\}$. For some set S, we know that if S is disjoint from $B^\circ(u, \rho_t)$ but intersects $B(v, \rho_{t-1})$, we have that $d(u, S) \geq \rho_t$ and $d(v, S) \leq \rho_{t-1}$, which implies that $|d(v, S) - d(u, S)| \geq \rho_t - \rho_{t-1}$.

By Lemma 21.17 we know that a set S has probability at least $c = \frac{1-e^{-1/4}}{2}$ of intersecting $B(v, \rho_{t-1})$ but being disjoint from $B^\circ(u, \rho_t)$. We can now randomly select our $O(\log^2 n)$ sets $S_1, \ldots, S_Q$ as in the book. From the proofs in the book, we know that the contribution of these sets is greater than or equal to $\frac{cd(u,v)}{4l}$ if and only if a sufficient percentage of the sets satisfy the above property (intersecting $B(v, \rho_{t-1})$ and being disjoint from $B^\circ(u, \rho_t)$). By Lemmas 21.20 and 21.21 we know this has a high probability of happening, and we also know from the proof of 21.21 that $O(\log n)$ sets will satisfy the property. Using this information, we see that

$$\begin{aligned}
\sum_{i=1}^{Q} |d(u,S_i) - d(v,S_i)| &\geq a \cdot \log n \cdot \sum_{i=1}^{\hat{t}+1} (\rho_i - \rho_{i-1}) \quad \text{for some constant } a \\
&= a \cdot \log n \cdot \rho_{\hat{t}+1} \quad \text{by Lemma 21.19} \\
&\geq \log n \cdot \frac{a \cdot d(u,v)}{2}.
\end{aligned}$$

Now, continuing with the hint, we use the fact that $|d(u, S_i) - d(v, S_i)| \leq d(u, v)$ to see that $f(u, v) \leq d(u, v)$. But we can also rearrange the above to see that

$$\begin{aligned}
d(u,v) &\leq \frac{2 \cdot \sum_{i=1}^{Q} |d(u,S_i) - d(v,S_i)|}{a \cdot \log n} \\
&= O\left(\log n \cdot \frac{\sum_{i=1}^{Q} |d(u,S_i) - d(v,S_i)|}{Q}\right) \quad \text{since } Q = O(\log^2 n) \\
&\leq O\left(\log n \cdot f(u,v)\right) \quad \text{by the monotonicity of the } \ell_p \text{ norm}
\end{aligned}$$

and so we have that $\frac{1}{O(\log n)} d(u, v) \leq f(u, v) \leq d(u, v)$ and we are done.