Assignment 11

Problem 26.12

Adding the additional constraint to MAX-CUT that we have two sets of pairs S1 and S2. $\forall(v_i, v_j) \in S1$ must be separated, and $\forall(v_i, v_j) \in S2$ must be together

We reduce the problem to the semidefinite program, and then we add the additional constraints on the semidefinite matrix A:
$\forall(i,j)s.t.(v_i, v_j) \in S1, A_{ij} = A_{ji} = -1$
$\forall(i,j)s.t.(v_i, v_j) \in S2, A_{ij} = A_{ji} = 1$
We then select a random hyperplane going through the space, and let each vertex v_i's placement be determined by which side of the hyperplane it's corresponding vector w_i is.

Lemma-The optimal solution to the SDP is at least as good to the optimal solution to the IP.
Taking the optimal solution to the integer program, we give every vertex on one side of the cut a vector $\hat{w}$, and every vertex on the other side of the cut a vector of $-\hat{w}$, where $\hat{w}$ is an arbitrary unit vector.
Taking the product of this matrix and it's transpose will produce a feasible semidefinite matrix, as it satisfies all of the original constraints of the MAX-CUT problem and also the additional constraints:

$\forall(i,j)s.t.(v_i, v_j) \in hpS1, A_{ij} = A_{ji} = \pm\hat{w} * \mp\hat{w} = -1$
$\forall(i,j)s.t.(v_i, v_j) \in S2, A_{ij} = A_{ji} = \pm\hat{w} * \pm\hat{w} = 1$
Therefore, we have used the optimal solution to the IP to produce a feasible solution to the SDP of equal utility, therefore showing that $OPT_{SDP} \geq OPT_{IP}$

Lemma:Using a random hyperplane cut on a feasible solution to the SDP will always produce a feasible solution to the IP.
Proof:When $(v_i, v_j) \in S1$ consider the angle θ between the two vectors w_i, w_j
$cos(\theta) = \frac{w_i * w_j}{|w_i||w_j|} = \frac{A_{ij}}{1} = -1$, $\theta = \pi$ Therefore, the vectors are antiparallel to each other, and there is no possible hyperplane that they could be put on the same side of, therefore satisfying this constraint.

Lemma:Using a random hyperplane cut on a feasible solution to the SDP will always produce a feasible solution to the IP. When $(i,j) \in S2$ consider the angle θ between the two vectors w_i, w_j
$cos(\theta) = \frac{w_i * w_j}{|w_i||w_j|} = \frac{A_{ij}}{1} = 1$, $\theta = 0$ Therefore, the vectors are parallel to each other, and there is no possible hyperplane that they could be put on the opposite sides of, therefore satisfying this constraint.

The hyperplane cutting method produces the same approximation factor, therefore, we have $AVSOLUTION \geq \alpha * OPT_{SDP} \geq \alpha * OPT_{IP}$, $\alpha \approx .878$, which is the factor we were looking for.