Problem Set 11

Problem 26.13

(The construction of the vectors is taken from the book.)

Consider the following k vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k \in \mathbb{R}^n$. Each vector has 0 in the last $n-k$ positions. Vector $\mathbf{x}_i$ has $-\sqrt{\frac{k-1}{k}}$ in the ith position and $1/\sqrt{k(k-1)}$ in the remaining positions.

If $G = (V, E)$ is k-colorable, then there exists a function $a : V \to \{1, 2, \ldots, k\}$ such that no adjacent vertices are assigned the same number. For each vertex $i \in V$, consider the assignment of the corresponding vector program variable $\mathbf{v}_i$ to $\mathbf{x}_{a(i)}$. In other words, set the vector program variable $\mathbf{v}_i$ equal to $\mathbf{x}_j$ if vertex i is assigned to color j. It remains to be shown that this assignment is feasible.

First, we show that the vectors $\mathbf{v}_i$ are norm 1. Since each $\mathbf{v}_i$ is equal to some $\mathbf{x}_k$, it suffices to show all $\mathbf{x}_i$ are norm 1, as demonstrated below.

$$\|\mathbf{x}_i\| = \left(-\sqrt{\frac{k-1}{k}}\right)^2 + (k-1)\left(\frac{1}{\sqrt{k(k-1)}}\right)^2 = \frac{k-1}{k} + (k-1)\left(\frac{1}{k(k-1)}\right) = 1$$

Second we show that for each edge $(i, j) \in E$,

$$\mathbf{v}_i \cdot \mathbf{v}_j \leq -\frac{1}{k-1}.$$

Since i and j are adjacent, we know $a(i) \neq a(j)$ (they were colored differently). Thus it must be that $\mathbf{v}_i \neq \mathbf{v}_j$. So we have

$$\begin{aligned}\mathbf{v}_i \cdot \mathbf{v}_j &= 2\left(-\sqrt{\frac{k-1}{k}}\right)\left(\frac{1}{\sqrt{k(k-1)}}\right) + (k-2)\left(\frac{1}{\sqrt{k(k-1)}}\right)^2 \\ &= -\frac{2}{k} + \frac{k-2}{k(k-1)} \leq -\frac{2}{k} + \frac{k-1}{k(k-1)} = -\frac{1}{k} < -\frac{1}{k-1}\end{aligned}$$