Problem Set 2

Problem 4.3

1. Since $f(u,v) \neq f(u,w)$, we have $f(u,v) < f(u,w) \leq f(v,w)$ (i.e., $f(u,v)$ is strictly less than the other two values). Consider a minimum u-v cut $(\mathcal{U},\mathcal{V})$, $u \in \mathcal{U}$, $v \in \mathcal{V}$. Because $(\mathcal{U},\mathcal{V})$ is a partition of V, w must be in either $\mathcal{U}$ or $\mathcal{V}$. Say that $w \in \mathcal{V}$. Then $(\mathcal{U},\mathcal{V})$ is also a u-w cut, but its value is strictly less than $f(u,w)$, a contradiction. The same argument holds if we say $w \in \mathcal{U}$ and compare the value of the cut to $f(v,w)$.

2. Assume that there are more than $n-1$ distinct values of $f(u,v)$. These define more than $n-1$ pairs of vertices, so if we connect each pair by an edge, there would necessarily be a cycle of edges. If the cycle has exactly three edges, then we have reached a contradiction to the statement proven in part 1, because there are three vertices whose pair-wise min-cut values are distinct. If the cycle has more than three edges, then consider two adjacent edges (w_1,w_2) and (w_2,w_3). Since $f(w_1,w_2)$ and $f(w_2,w_3)$ are distinct, it must be that $f(w_1,w_3)$ is equal to one or the other of them (from part 1). Therefore we can remove (w_1,w_2) and (w_2,w_3) from the cycle and replace them with (w_1,w_3), while still maintaining the property that the endpoints of all edges in the cycle have distinct min-cut values. Continue this process until the cycle has only 3 edges, and we have again reached a contradiction.

3. As shown in part 1, the smaller two numbers in $\{f(u,v), f(u,w), f(w,v)\}$ are equal. So we need to consider the cases that $f(u,v)$ is a member of that smaller pair, or that it isn't. If it is, then $\{f(u,w), f(w,v)\}$ must include the other member of the smaller pair, and thus $\min\{f(u,w), f(w,v)\} = f(u,v)$. If it isn't, then $\{f(u,w), f(w,v)\}$ are themselves the smaller pair, and we have $\min\{f(u,w), f(w,v)\} = f(u,w) \leq f(u,v)$.

4. The proof is by induction. From part 3, we have $f(w_k,v) \geq \min\{f(w_k,w_{k+1}), f(w_{k+1},v)\}$, so we know that:

$$\begin{aligned} & f(u,v) \geq \min\{f(u,w_1), f(w_1,w_2), \ldots, f(w_{k-1},w_k), f(w_k,v)\} \\ \Rightarrow\ & f(u,v) \geq \min\{f(u,w_1), f(w_1,w_2), \ldots, f(w_{k-1},w_k), \min\{f(w_k,w_{k+1}), f(w_{k+1},v)\}\} \\ & \qquad \geq \min\{f(u,w_1), f(w_1,w_2), \ldots, f(w_k,w_{k+1}), f(w_{k+1},v)\} \end{aligned}$$

   We also have the base case $f(u,v) \geq \min\{f(u,w_1), f(w_1,v)\}$, so the statement is proven by induction over $k = 1, 2, \ldots, r$.