Approximation Algorithms

Homework #3

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Exercise 5.4

Lemma 1: Given a graph H, let I be an independent set in H^2. Then, $\alpha|I| \leq dom_\alpha(H)$.

Proof: Let D be a minimum α-dominating set in H. Each vertex in D is connected to at least $(\alpha - 1)$ other vertices from D. Thus for every α vertices from D we have a clique in H^2 , i.e. $\frac{|D|}{\alpha}$ number of cliques. We can pick at most one vertex from each clique for independent set I. Hence, $|I| \leq \frac{|D|}{\alpha}$, the lemma follows.

Algorithm: (Fault-tolerant k-center problem)

1. Construct $G_1^2, G_2^2, \ldots, G_m^2$.
2. Compute a maximal independent set M_i in each graph G_i^2.
3. Find the smallest index i such that $|M_i| \leq \lfloor \frac{k}{\alpha} \rfloor$ and every vertex in M_i has a degree of at least $\alpha - 1$ in G_i.
4. Pick $\alpha - 1$ vertices that each vertex from M_i is adjacent to in G_i, call it $M\alpha_i$. Return $M\alpha_i \cup M_i$.

   Note that when $k > \alpha \times |M_i|$, one of the vertices in M_i will have a degree of at least $(k \bmod \alpha) - 1$. For this vertex we output $(k \bmod \alpha) - 1$ of its adjacent vertices instead of only $\alpha - 1$.

Lemma 2: For j as defined in the algorithm, cost$(e_j) \leq OPT$.

Proof: For every $i < j$ we have that $|M_i| > \lfloor \frac{k}{\alpha} \rfloor$. By the above Lemma, $dom_\alpha(H) > \lfloor \frac{k}{\alpha} \rfloor$, and so $i^* > i$, where i^* is the index of the most expensive edge in an optimal fault-tolerant k-center solution. Hence, $j \leq i^*$.

Theorem: Above algorithm achieves an approximation of factor 3 for fault tolerant k-center problem.

Proof: From the proof of Theorem 5.5 in Vazirani textbook we know that after step 3 of the above algorithm there exist stars in G_j^2 centered on the vertices of M_j, covering all vertices. Let us consider one of these stars that contains a set of vertices W. Consider the following partition of vertices in this star: vertex v, such that $v \in M_j, v \in W$, set $C = \{v_c | v_c \in M\alpha_j, v_c \in W\}$, and set $O = W \setminus (\{v\} \cup C)$. The cost of edges from each vertex $v_c \in C$ to v is at most $\text{cost}(e_j)$ as for $M\alpha_j$ we pick only vertices that are adjacent to vertices from M_j in G_j (by the algorithm we have not yet picked any edge of cost $> \text{cost}(e_j)$).

By Theorem 5.5 we also know that each edge used in construction of these stars has a cost of at most $2 \cdot \text{cost}(e_j)$. Hence, by the triangle inequality, the distance from each vertex $v_o \in O$ to $(\alpha - 1)$ vertices from C will be at most the cost of path to v, $2 \cdot \text{cost}(e_j)$, and cost of a path from v to $(\alpha - 1)$ v_c vertices of cost $\leq \text{cost}(e_j)$. Hence a distance from v_o to v and $(\alpha - 1)$ of vertices in C is $\leq 3 \cdot \text{cost}(e_j)$. The theorem follows from Lemma 2.

By the triangle inequality the cost of a distance between any two vertices v_{c_l} and v_{c_m} from set C is at most $2 \cdot \text{cost}(e_j)$, i.e. distances to v from both of them.

The above analysis holds for any star in G_j^2. □