Approximation Algorithms
Homework 4
Yuri Malitsky

12.1

Show that the dual of the dual of a linear program is the original program itself.

Without loss of generality, we shall assume that our primal program is a minimization problem in standard form:

$$\begin{array}{lll} \text{minimize} & \sum_{j=1}^{n} c_j x_j & \\ \text{subject to} & \sum_{j=1}^{n} a_{ij} x_j \geq b_i & i = 1, ..., m \\ & x_j \geq 0 & j = 1, ..., n \end{array}$$

As is shown in the book, the dual of this problem is:

$$\begin{array}{lll} \text{maximize} & \sum_{i=1}^{m} b_i y_i & \\ \text{subject to} & \sum_{i=1}^{m} a_{ij} y_i \leq c_j & j = 1, ..., n \\ & y_i \geq 0 & i = 1, ..., m \end{array}$$

This dual form is a maximization problem, thus we convert it to a more familiar minimization problem by negating its objective:

$$\begin{array}{lll} \text{minimize} & \sum_{i=1}^{m} -b_i y_i & \\ \text{subject to} & \sum_{i=1}^{m} a_{ij} y_i \leq c_j & j = 1, ..., n \\ & y_i \geq 0 & i = 1, ..., m \end{array}$$

The dual problem however is still subject to $\leq$ constraints instead of the desired $\geq$. To fix this we multiply the constraints by -1.

$$\begin{array}{lll} \text{minimize} & \sum_{i=1}^{m} -b_i y_i & \\ \text{subject to} & \sum_{i=1}^{m} -a_{ij} y_i \geq -c_j & j = 1, ..., n \\ & y_i \geq 0 & i = 1, ..., m \end{array}$$

Now the dual problem is in standard form and we can easily take the dual of it:

$$\begin{array}{lll} \text{maximize} & \sum_{j=1}^{m} -c_j z_j & \\ \text{subject to} & \sum_{j=1}^{n} -a_{ij} z_j \leq -b_i & i = 1, ..., m \\ & z_i \geq 0 & j = 1, ..., n \end{array}$$

Again, the dual of the dual in standard form is a maximization problem with

$\leq$ constraints, thus to convert the problem to the familiar standard form, we multiply both the objective and the constraints by -1.

$$\begin{array}{lll} \text{minimize} & \sum_{j=1}^{m} c_j z_j & \\ \text{subject to} & \sum_{j=1}^{n} a_{ij} z_j \geq b_i & i = 1, ..., m \\ & z_j \geq 0 & j = 1, ..., n \end{array}$$

And after this conversion we see that the dual of the dual is identical to our original primal program with z and x being aliases to the same variable vectors.