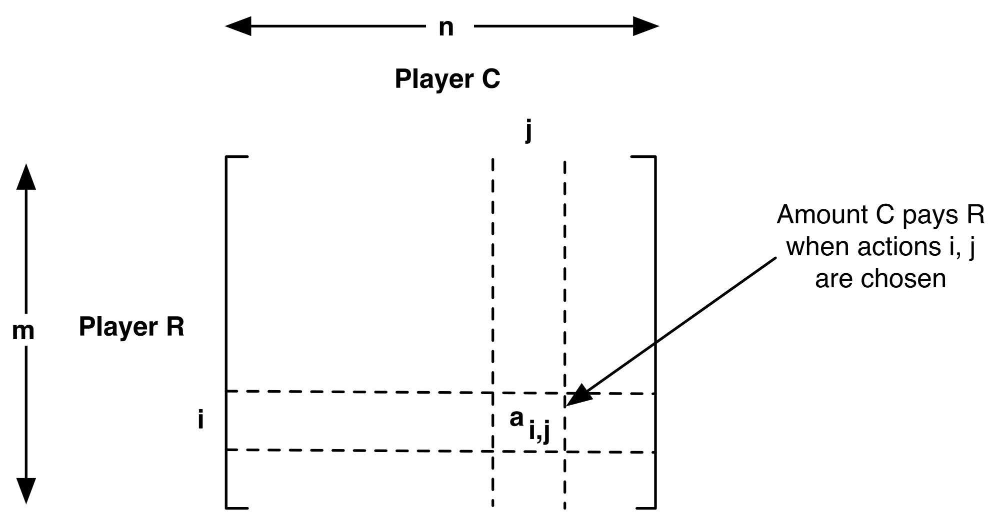
Homework 4

Problem 12.11

In this problem, we derive the minimax theorem using the LP-duality theorem. We are given a finite two-person zero-sum game, represented by matrix A. Player R can play actions from $\{1, ..., m\}$, and player C can play actions from $\{1, ..., n\}$.



Each player R and C will choose a strategy, $\mathbf{x}$ and $\mathbf{y}$ respectively, which is a vector that represents the probability of playing the action in each row or column.

$$\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}, \mathbf{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \tag{1}$$

Find the dual of the LP computing R's optimal strategy

We are given an LP Z that computes the optimal strategy for player R:

$$\begin{array}{lll} maximize & z & \\ s.t. & z - \sum_{i=1}^{m} a_{ij} x_i \leq 0 & j = 1, ..., n \\ & \sum_{i=1}^{m} x_i = 1 & \\ & x_i \geq 0, & i = 1, ..., m \end{array}$$

The dual of LP Z is:

minimize	w	
s.t.	$w - \sum_{j=1}^{n} a_{ij}y_j \geq 0$	$i = 1, ..., m$
	$\sum_{j=1}^{n} y_j = 1$	
	$y_j \geq 0,$	$j = 1, ..., n$

Show that the dual of Z computes the optimal strategy for C

The optimal strategy for player C is the one that guarantees maximum possible expected winnings (or minimum expected losses), regardless of the strategy chosen by the other player. If player C plays strategy $\mathbf{y}$, he or she can be sure of losing no more than $\max_{\mathbf{x}} \mathbf{x}^{\mathbf{T}}\mathbf{A}\mathbf{y}$, where the maximum is taken over all of player R's possible strategies. A strategy $\mathbf{y}$ that minimizes this value is an optimal strategy for player C.

We know that at least one of the optimal responses to player C's strategy is a pure strategy:

$$\max_{\mathbf{x}} \mathbf{x}^{\mathbf{T}}\mathbf{A}\mathbf{y} = \max_{\mathbf{i}} \sum_{\mathbf{j=1}}^{\mathbf{n}} \mathbf{a_{ij}y_j} \tag{2}$$

The goal is to find a strategy $\mathbf{y}$ for player C that will minimize this value. This can be expressed as an optimization problem:

minimize	$\max_i \sum_{j=1}^{n} a_{ij}y_j$	
s.t.	$\sum_{j=1}^{n} y_j = 1$	
	$y_j \geq 0,$	$(j = 1, ..., n)$

Expressed as an LP (call it W), the problem of computing player C's optimal strategy takes the form:

minimize	w	
s.t.	$w - \sum_{j=1}^{n} a_{ij}y_j \geq 0$	$i = 1, ..., m$
	$\sum_{j=1}^{n} y_j = 1$	
	$y_j \geq 0,$	$j = 1, ..., n$

This LP (which computes C's optimal strategy) is exactly the same as the dual of Z. Thus, the dual of Z computes the optimal strategy for C.

Prove the minimax theorem using the LP-duality theorem

We first state the minimax theorem and the LP-duality theorem.

Minimax theorem: For every matrix $\mathbf{A}$,

$$\max_{\mathbf{x}} \min_{\mathbf{y}} \mathbf{x}^{\mathbf{T}}\mathbf{A}\mathbf{y} = \min_{\mathbf{y}} \max_{\mathbf{x}} \mathbf{x}^{\mathbf{T}}\mathbf{A}\mathbf{y} \tag{3}$$

LP-duality theorem: The primal program has finite optimum iff its dual has finite optimum. Moreover, if $\mathbf{x}^* = (\mathbf{x_1^*}, ..., \mathbf{x_n^*})$ and $\mathbf{y}^* = (\mathbf{y_1^*}, ..., \mathbf{y_m^*})$ are optimal solutions for the primal and dual programs, respectively, then

$$\sum_{j=1}^{n} c_j x_j^* = \sum_{i=1}^{m} b_i y_i^* \tag{4}$$

From the LP-duality theorem, we know that the optimal solution of the primal is equal to the optimal solution to the dual.

We also have an LP Z that computes the optimal strategy for R, and we have shown that its dual is an LP W that computes the optimal strategy for C.

Putting these together, we know that the optimal solution of Z is equal to the optimal solution of W. Our Z is equal to $\max_{\mathbf{x}} \min_{\mathbf{y}} \mathbf{x^T A y}$, and our W is equal to $\min_{\mathbf{y}} \max_{\mathbf{x}} \mathbf{x^T A y}$, so we find that:

$$\max_{\mathbf{x}} \min_{\mathbf{y}} \mathbf{x^T A y} = \min_{\mathbf{y}} \max_{\mathbf{x}} \mathbf{x^T A y} \tag{5}$$

This gives us the minimax theorem. Thus, the minimax theorem is true.