Approximation Algorithms
Homework #4

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Exercise 12.8

König-Egerváry Theorem: In any bipartite graph,

$$\max |M| = \min |U|,$$

where M is a matching and U is a vertex cover for G.

Proof via LP-duality: Let $G = (V, E)$ be a bipartite graph. Following the steps as suggested in Vazirani.

(1) Consider the following LP-problem

$$\begin{aligned} &\text{maximize} \sum_e x_e \\ &\text{subject to} \sum_{e:\ e \text{ is incident at } v} x_e \le 1, \quad v \in V \\ &x_e \ge 0, \quad e \in E \end{aligned} \tag{1}$$

Claim: LP(1) is an exact LP-relaxation for maximum matching problem in G.

Proof: If we restrict $x_e \in \{0, 1\}$ in LP(1) we will obtain an LP problem that seeks 0/1 solutions.

$$\begin{aligned} &\text{maximize} \sum_e x_e \\ &\text{subject to} \sum_{e:\ e \text{ is incident at } v} x_e \le 1, \quad v \in V \\ &x_e \in \{0, 1\} \quad e \in E \end{aligned} \tag{2}$$

LP(2) restricts that for each vertex v we have at most one variable x_e assigned to 1. Variables x_e that are set to 1 are part of a matching in G: if $x_e = 1$ then edge e is in the matching, as the constraint restricts that there is at most one edge e for each vertex. As the objective is to maximise the sum of variables x_e, i.e. the number of x_e that are set to 1, then the optimum

solution is a maximal matching in graph G.
To show that LP(1) is the exact LP-relaxation for LP(2) we construct a constraint matrix A for LP(1), such that every column represents an edge e and row represents a vertex v in G. Each entry (e, v) in A is either 1, if edge e is incident with vertex v, and 0 otherwise.
Claim: Matrix A is totally unimodular, i.e. every square submatrix of A has a determinant of -1, 0, or 1.
Proof: Each row in A contains only 0s or 1s. We will show that each submatrix in A has a determinant of 0, -1, or 1 by induction.

$\mathbf{n = 2}$ If size of a subgraph is 2×2 then we can only have the following combinations:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

or their reflections (assuming there is only one edge between every two vertices). Hence, $\det(A)$ is either 0, 1, -1.

$\mathbf{n = k}$ Consider square submatrix A_k of size $k \times k$. Suppose all submatrices of size $k - 1$ have a determinant of either 0, -1 or 1. Each column in matrix A can contain at most two 1's, as each edge is connected to two vertices, thus A_k will have at most two 1's in each column. We will consider each case separately.

When A_k has a column with all 0s, $\det(A_k) = 0$.

When A_k has a column with one 1 entry we can pick this column and calculate the determinant of the corresponding minor, call it M. The determinant of A_k will then be either positive or negative of $\det(M)$. By hypotheses $\det(M)$ will be 0, -1, or 1.

Now let us consider the case when A_k contains two 1 entries in each column. We can divide the rows in A_k in two partitions, rows for the vertices in one partition of bipartite graph and another one. If we sum the rows from one partition and compare to the sum of the row factor for another partition, we will obtain the identical row factors. The later is due to the fact that vertices in the same partition of bipartite graph do not have any edge in common. Rows in A_k are linearly dependant and hence $\det(A_k) = 0$ [1]. Hence determinant of every square subgraph in A is unimodular.

The constraint matrix for the above problem is totally unimodular and from Exercise 12.7 we can conclude that any solution to LP(1) will always be integral.

[1] The main idea of the above proof was inspired from http://www2.math.uni-paderborn.de/fileadmin/Mathematik/AG-Eisenbrand/teaching/opt2006/19-12.pdf

(2) Let us construct a dual problem of LP(2). We introduce new variables v_k corresponding to constraints in LP(2).

$$\begin{aligned} &\text{minimize} \sum_{v_k \in V} v_k \\ &\text{subject to} \sum_{v_i, v_j \text{vertices of } e} v_i + v_j \geq 1, \quad e \in E \\ &v_k \geq 0, \quad v_k \in V \end{aligned} \tag{3}$$

Claim: The above dual LP-problem is an exact LP-relaxation for minimum vertex cover in bipartite graph G.

Proof: If we restrict $v_k \in \{0, 1\}$ we will obtain an LP problem that seeks 0/1 solutions.

$$\begin{aligned} &\text{minimize} \sum_{v_k \in V} v_k \\ &\text{subject to} \sum_{v_i, v_j \text{vertices of } e} v_i + v_j \geq 1, \quad e \in E \\ &v_k \in \{0, 1\}, \quad v_k \in V \end{aligned} \tag{4}$$

LP(4) problem restricts that at least one of v_i, v_j variables for each edge in G has to be set to 1. This restriction is achieved via first constraint. As minimising the sum of v_k variables is the objective, the set of variables that are set to 1 in the optimum solution represents the minimum vertex set cover, i.e. if $v_k = 1$ then vertex v_k is in the minimum vertex set cover.

LP(3) is an exact LP-relaxation of LP(4). The constraint matrix for LP(3) is a transpose matrix for the constraint matrix that we constructed for LP(1), and is totally unimodular as well. LP(3) gives only integral solutions, and hence is an exact LP-relaxation of LP(4).

(3) From LP-duality theorem the number of edges in a maximum matching set must be equal to the number of vertices in at minimum vertex set cover. □