CSCI2510 Approximation Algorithms
Assignment 4

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Exercise 12.9

1. Let $G = (V, E)$ be an undirected graph, with weights w_e on edges, and let $\mathcal{O} = \{S \subset V : |S| odd\}$. (By $e : e \in S$ we mean edges e that have both endpoints in S.)

$$\begin{array}{lll} \text{maximize} & \displaystyle\sum_{e \in E} w_e x_e & \\ \text{subject to} & \displaystyle\sum_{e:e \text{ incident at } v} x_e \leq 1, & v \in V \\ & \displaystyle\sum_{e:e \in S} x_e \leq \frac{|S|-1}{2}, & S \in \mathcal{O} \\ & x_e \geq 0, & e \in E \end{array}$$

Fact 1. *The program above is an exact LP-relaxation for the problem of finding a maximum weight matching in G.*

The dual of the above LP includes one variable y_v for every vertex $v \in V$ and one variable y_S for every set $S \in \mathcal{O}$ (the two different types of variables correspond to the two different type of constraint in the primal LP):

$$\begin{array}{lll} \text{minimize} & \displaystyle\sum_{v \in V} y_v + \sum_{S \in \mathcal{O}} \frac{|S|-1}{2} y_S & \\ \text{subject to} & \displaystyle y_u + y_v + \sum_{S:e \in S} y_S \geq w_e, & e = \{u, v\} \in E \\ & y_v \geq 0, & v \in V \\ & y_S \geq 0, & S \in \mathcal{O} \end{array}$$

Fact 2. *If the weight function is integral, the dual is also an exact LP-relaxation.*

Definition 1. *An odd set cover in a graph $G = (V, E)$ is a pair (C, D), in which C is a collection $v_1, \ldots, v_l$ of vertices and D is a collection $S_1, \ldots, S_k$ of disjoint odd cardinality subsets of V such that each edge of G is either incident at one of the vertices v_i or has both endpoints in one of the sets S_i. The weight w of the odd set cover is $l + \sum_{i=1}^{k}(|S_i| - 1)/2$.*

Theorem 1. *In any graph,* $\max_{\text{matching } M} |M| = \min_{\text{odd set cover } OC} w(OC)$.

Proof. From Fact 1, we can find $\max_{matching M} |M|$ from the LP-relaxation with weights $w_e = 1, \forall e \in E$. With this choice of the weights, we have that also the dual is an exact LP-relaxation, thus the optimal solution of the dual is integral and the value of the optimal solution in the primal and the dual is the same (LP-duality theorem). Given a solution y of the dual LP, consider the sets $C = \{v : y_v = 1\}$ and $D = \{S : y_S = 1\}$. If (C, D) is an odd set cover, we have that the cost of the solution y is the weight of (C, D).

To prove the theorem, we thus need to show:

(a) each odd set cover corresponds to a solution of the dual LP with integer constraints on $y_v, v \in V$ and $y_S, S \in \mathcal{O}$;

(b) the optimal solution of the dual corresponds to an odd set cover.

Note: it is not true that each solution to the dual LP with integer $\{0, 1\}$ *constraints is equivalent to an odd set cover, because in the dual LP there is not constraint on the disjointness of the sets in* D.

Consider a odd set cover for the graph and define the following assignment for the LP:

$$y_v = 1 \text{ if } v \in C, 0 \text{ otherwise;}$$

$$y_S = 1 \text{ if } S \in D, 0 \text{ otherwise;}$$

Since for each edge $e = \{u, v\} \in E$ we have that it is incident to one of the vertices in C or has both endpoints in one of the sets $S_i \in D$, we have $y_u + y_v + \sum_{S:e \in S} y_S \geq 1 = w_e$, thus this is a feasible solution of the dual LP; this proves (a).

To prove (b), consider the optimal solution y^* of the dual LP of cost OPT; we first prove that $y^*_v \in \{0, 1\}, \forall v \in V$ and $y^*_S \in \{0, 1\}, \forall S \in \mathcal{O}$. The proof is by contradiction: let assume that there exist $y^*_v \notin \{0, 1\}$; now consider the assignment obtained from y^* setting $y^*_v = 1$, of cost $<$ OPT. This solution is feasible because $\forall e = \{u, v\} \in E$ we have $y_u + y_v + \sum_{S:e \in S} y_S \geq 1 = w_e$, thus there exist a feasible solution of cost $<$ OPT, that is a contradiction. The same reasoning gives $y^*_S \in \{0, 1\}$.

Now consider the set of vertices $C = \{v : y^*_v = 1\}$ and the set of subsets $D = \{S \subset V : y^*_S = 1\}$. Given the constraint of the dual on the edges, we have that each edge in G is either incident at one of the vertices in C or has both endpoints in one of the sets S_i; thus if (C, D) does to not constitute an odd set cover, there are non-disjoint sets in D. We will show that in this case y^* either is not the optimal solution or we can find an optimal solution with the same cost y^* in which all the sets of D are disjoint.

Suppose that S_1, S_2 in D are not disjoint, and denote their intersection with I; let $|S_1| = 2s_1 + 1, |S_2| = 2s_2 + 1, |I| = c > 0$. We differentiate the analysis for c even and c odd. Let suppose that c is even: if we change y^* setting $y_{S_1} = y_{S_2} = 0$, $y_{S_1 \setminus I} = y_{S_2 \setminus I} = 1$, and $y_v = 1 \forall v \in I$, we have a feasible solution and the value of this solution is $\text{OPT} - \frac{2\text{s}_1}{2} - \frac{2\text{s}_2}{2} + \frac{2\text{s}_1 - \text{c}}{2} + \frac{2\text{s}_2 - \text{c}}{2} + \text{c} = \text{OPT}$, thus we can find an optimal solution of the dual that corresponds to an odd set cover. Let now consider c odd: if we consider the assignment obtained by y^* setting $y_{S_1} = y_{S_2} = 0$, $y_{S_1 \cup S_2} = 1$, we obtain a feasible solution of value $\mathrm{OPT} - \frac{2s_1}{2} - \frac{2s_1}{2} + \frac{2s_1+1+2s_2+1-c+1}{2} = \mathrm{OPT} + \frac{1-c}{2}$. If $c = 1$, we obtain an optimal solution that corresponds to an odd set cover; if $c > 1$, we obtain a feasible solution of cost $< \mathrm{OPT}$, that is a contradiction. □

2. Assume that $|V|$ is even.

Fact 3. *The following is an exact LP-relaxation for the minimum weight perfect matching problem in G:*

$$\begin{aligned}
&\textit{minimize} && \sum_{e \in E} w_e x_e \\
&\textit{subject to} && \sum_{e: e \text{ incident at } v} x_e = 1, && v \in V \\
& && \sum_{e: e \in S} x_e \leq \frac{|S|-1}{2}, && S \in \mathcal{O} \\
& && x_e \geq 0, && e \in E
\end{aligned}$$

Note that this LP is not in standard form due to equality constraints.

To find the dual LP, we assign a multiplier y_v to every constraint on $v \in V$ in the primal, and a multiplier y_S to each constraint on $S \in \mathcal{O}$; then the steps to find the dual LP are the same used to find the dual of a primal LP in standard form, but since the equality constraints are preserved even when multiplied by a negative number, the variables y_v have not constraints on the sign. The dual LP is then:

$$\begin{aligned}
&\text{maximize} && \sum_{v \in V} y_v - \sum_{S \in \mathcal{O}} \frac{|S|-1}{2} y_S \\
&\text{subject to} && y_u + y_v - \sum_{S: e \in S} y_S \leq w_e, && e = \{u, v\} \in E \\
& && y_S \geq 0, && S \in \mathcal{O}
\end{aligned}$$

Using complementary slackness conditions, we obtain the following conditions for a pair x^*, y^* of optimal primal and dual solutions:

$$\forall v \in V, \quad y_v^* = 0 \text{ or } \sum_{e: e \text{ incident at v}} x_e^* = 1 \tag{1}$$

$$\forall S \in \mathcal{O}, \quad y_S^* = 0 \text{ or } \sum_{e: e \in S} x_e^* = \frac{|S|-1}{2} \tag{2}$$

$$\forall e \in E, \quad x_e^* = 0 \text{ or } y_u^* + y_v^* - \sum_{S: e \in S} y_S^* = w_e \tag{3}$$

However, given the equality constraints in the primal, we have that conditions (1) are redundant, since we already know that $\sum_{e: e \text{ incident at } v} x_e^* = 1, \forall v \in V$.

Remark: the dual of LP can be obtained also substituting the set of constraints

$$\sum_{e:e \text{ incident at } v} x_e = 1, v \in V$$

in the primal LP with this two sets of constraints:

$$\sum_{e:e \text{ incident at } v} x_e \leq 1, v \in V$$
$$\sum_{e:e \text{ incident at } v} x_e \geq 1, v \in V$$

With this substitution, in the dual LP for each $v \in V$ there are two variables y_v^L, y_v^G corresponding to the set of "≤ 1" constraints and to the set of "≥ 1" constraints, respectively, with the additional constraints $y_v^L \geq 0, y_v^G \geq 0$. This corresponds to multiply the constraint $\sum_{e:e \text{ incident at } v} x_e = 1$ for $y_v^G - y_v^L$, that is exactly what was done in the solution of the exercise. Using the complementary slackness conditions in this case, we have no additional constraints on y_v^G, y_v^L, and thus there are no constraints on the sign of $y_v^G - y_v^L$.