Problem Set 4

Problem 6.2

Consider an instance of vertex cover $G = (V, E)$. Construct the graph $G' = (V', E')$ and the weight function $w : V' \to \mathbb{R}$, such that: V' contains V plus an additional vertex v_e for each edge $e \in E$; the set E' contains each of the edges $e = (u_1, u_2) \in E$ plus additional edges (v_e, u_1) and (v_e, u_2); and finally, $w(u) = 1$ for each $u \in V$, $w(v_e) = \infty$ for each $e \in E$. Thus each original edge e is part of a 3-cycle C_e defined by the vertices $\{u_1, u_2, v_e\}$.

The claim is that any α-approximation S of minimum-weight feedback vertex set on (G', w) is an α-approximation of the original instance of vertex cover on G.

Lemma 1. *Any α-approximation S of feedback vertex set on (G', w) is a subset of the original vertices V.*

Proof. It suffices to show that any α-approximation on (G', w) must have finite weight. The set V itself is clearly a feedback vertex set on G', and its weight is $|V|$. Since the weight of FVS_{opt} is bounded above by that quantity, it must be finite. Thus for finite α, any α-approximation must have finite weight. □

Lemma 2. *A set of vertices $S \subseteq V$ is a feedback vertex set on G' if and only if S is a vertex cover on G.*

Proof. If we assume S is a feedback vertex set on G', then for each $e = (u_1, u_2) \in E$, either $u_1 \in S$ or $u_2 \in S$ (otherwise the cycle C_e would remain unbroken). Thus S is a vertex cover on G. If we assume S is a vertex cover on G, then since every cycle of edges in E' contains at least one edge in E, every such cycle will be broken by an endpoint of one such edge. □

The lemmas, combined with the fact that the weight of any set $S \subseteq V$ is equal to $|S|$ under both objective functions proves the reduction.