Approx. Algo. HW5

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1 Weighted set cover problem

`Input:` A finite set $S = \{1, .., n\}$ and a collection $C = \{C_1, .., C_m\}$ of subsets of S and a weight function $w : C \to \mathbf{R}$.

`Output:` A subset $C' \subseteq C$ such that $\cup_{c \in C'} c = S$.

`Objective:` Minimize $\sum_{c \in C'} w(c)$.

`IP model:` $X_i \in \{0, 1\}$ is a variable corresponds to $C_i \in C$, $X_i = 1$ if $C_i \in C'$, 0 otherwise.

$Y_{ij} \in \{0, 1\}$ is a variable corresponds to element j in C_i. $Y_{ij} = 1$ if $j \in C_i$ and 0 otherwise.

$$\begin{aligned}
\min \sum_{i=1}^{n} w(X_i) \quad & \text{s.t.} & & \quad (1)\\
|C_i| X_i &= \sum_{j \in C_i} Y_{ij} \;\; i \in \{1, .., n\} & & \quad (2)\\
Y_{ij} &= 0 \;\; i \in \{1, .., n\}, j \notin C_i & & \quad (3)\\
\sum_{j=1}^{n} Y_{ij} &\geq 1 \;\; i \in \{1, .., n\} & & \quad (4)\\
X_i &\in \{0, 1\} \;\; i \in \{1, .., n\} & & \quad (5)\\
Y_{ij} &\in \{0, 1\} \;\; i \in \{1, .., n\}, j \in \{1, .., m\} & & \quad (6)
\end{aligned}$$

Line 2 forces that when C_i is chosen ($X_i = 1$), all variables Y_{ij} corresponds to the values $j \in C_i$ are set to 1, which means they are covered. Line 3 guarantees that if value j is not in a set C_i, it will be zero. Line 4 requires a value j should be covered by at least one set ($Y_{ij} = 1$).

2 MST

`Input:` An Undirected Graph $G = (V, E)$ and a weight function $w : E \to \mathbf{R}$.

`Output:` A Tree $T = (V, E_T)$ such that $E_T \subseteq E$.

`Objective:` Minimize $\sum_{e \in E_T} w(e)$.

`IP Model:` $X_e \in \{0, 1\}$ is a variable correspond to an edge $e \in E$. $X_e = 1$ if $e \in E_T$, 0 otherwise.

$$\begin{aligned}
\min \sum_{e \in E} w(X_e) \quad \text{s.t.} & & \qquad (7)\\
\sum_{e \in E} X_e &= |V| - 1 & \qquad (8)\\
\sum_{e \in E'} X_e &\leq |V'| - 1 \;\; G = (V', E') \text{ is a induced subgraph of} G & \qquad (9)\\
X_e &\in \{0, 1\} \;\; e \in E & \qquad (10)
\end{aligned}$$

Line 8 guarantees the output graph has $|V|-1$ edges, the necessary condition for a tree. Line 9 guarantees there is no cycle in the output graph, by enumerating all possible induced subgraph of G and make sure they cannot form any cycle, since whenever there is a cycle $C = (V_C, E_C)$ in G', $\sum_{e \in E_C} X_e = |V_C|$. Since there is no cycle in T, and T has $|V| - 1$ edges, T is a tree.

3 TSP

`Input:` An undirected graph $G = (V, E)$ and a weight function $w : E \rightarrow \mathbf{R}$.

`Output:` A tour E' such that $E' \subseteq E$ and each vertex $v \in V$ is visited exactly once.

`Objective:` Minimize $\sum_{e \in E'} w(e)$.

`IP Model:` $X_e \in \{0, 1\}$ is a variable correspond to an edge $e \in E$. $X_e = 1$ if $e \in E'$, 0 otherwise.

$$\begin{aligned}
\min \sum_{e \in E} w(X_e) \quad \text{s.t.} & & \qquad (11)\\
\sum_{e \in E} X_e &= |V| & \qquad (12)\\
\sum_{e \in E'} X_e &\leq |V'| - 1 \;\; (V', E') \text{ is a induced proper subgraph of} G & \qquad (13)\\
X_e &\in \{0, 1\} \;\; e \in E & \qquad (14)
\end{aligned}$$

Line 12 guarantees the output graph has $|V|$ edges, a necessary condition for a tour. Line 13 guarantees there is no cycle in the output graph except the tour itself, by enumerating all possible proper induced subgraph of G and make sure they cannot form any cycle, since whenever there is a cycle $C = (V_C, E_C)$ in G', $\sum_{e \in E_C} X_e = |V_C|$.

4 Feedback Vertex Set

`Input:` An undirected graph $G = (V, E)$ and a weight function $w : V \to \mathbf{R}$.

`Output:` A set $V' \subseteq V$ such that there is no cycle in the induced subgraph $G[V \setminus V']$, for simplicity we call it $G_{-V'}$.

`Objective:` Minimize $\sum_{v \in V'} w(v)$.

`IP Model:` X_v is a variable corresponds to a vertex, it is 1 if $v \in V'$, 0 if otherwise.

Y_e is a variable corresponds to an edge, it is 1 if $e \notin G_{-V'}$, 0 if otherwise.

$d(v)$ denotes the degree of vertex v in G.

$$\begin{aligned}
d(v)X_v &= \sum_{e \text{ adj to } v} Y_e \quad v \in V && (15)\\
\sum_{e \in G'} (1 - Y_e) &\leq |V'| - 1 \quad G' = (V', E') \text{ is a induced subgraph of } G && (16)\\
X_v &\in \{0, 1\} \quad v \in V && (17)\\
Y_e &\in \{0, 1\} \quad e \in E && (18)\\
& && (19)
\end{aligned}$$

If v is in V', all of its adjacent edges are removed, i.e. $Y_e = 1$.

$1 - Y_e = 1$ if edge e is not removed, line 16 guarantees that there is no cycle in any induced subgraph of G.

5 s-t shortest path

`Input:` An directed graph $G = (V, E)$, two vertices $s, t \in V$ and a weight function $w : E \to \mathbf{R}$.

`Output:` A simple path P in G starts from s and ends at t.

`Objective:` Minimize $\sum_{e \in P} w(e)$.

`IP Model:` $X^+_{(u,v)}$ indicates the status of the edge coming out from u and reach v, it is 1 if $(u, v) \in P$ and 0 if otherwise.

$X^-_{(u,v)}$ indicates the status of the edge going in to v from u, it is 1 if $(u, v) \in P$ and 0 if otherwise.

$$\min \sum_{e \in E} w(X_e) \quad \text{s.t.} \tag{20}$$

$$\sum_{(v,u) \in E} X^+_{(v,u)} = \sum_{(u,v) \in E} X^-_{(u,v)} \quad u \in V \setminus \{s, t\} \tag{21}$$

$$\sum_{(v,t) \in E} X^+_{(v,t)} = 1 \tag{22}$$

$$\sum_{(s,v)\in E} X^-_{(s,v)} = 1 \tag{23}$$

$$X^+_{(u,v)}, X^-_{(u,v)} \in \{0,1\} \ \ (u,v)\in E \tag{24}$$

This model is similar to the model of maximum flow algorithm. Here we restrict that only one unit of flow from s to t, and that flow is actually the path P.