

## 5

First, let's write this in standard form:

Maximize  $y_{ws} + 2y_{mar} + 2y_{man} + 3y_{pub}$  subject to:

$$y_{ws} + y_{man} + y_{pub} \leq 32$$

$$y_{man} + y_{mar} \leq 32$$

$$y_{mar} + y_{pub} \leq 32$$

$$y_{ws}, y_{mar}, y_{man}, y_{pub} \geq 0$$

where  $y_{ws}$  represents the number of whiskey sours,  $y_{mar}$  represents the number of martinis, etc. We can also introduce slack variables  $x_{2w}$ ,  $x_v$ , and  $x_{2g}$  (these will represent the amount of each kind of liquor remaining) such that

$$x_{2w} = 32 - y_{ws} - y_{man} - y_{pub} \quad (1)$$

$$x_v = 32 - y_{man} - y_{mar} \quad (2)$$

$$x_{2g} = 32 - y_{mar} - y_{pub} \quad (3)$$

$$x_{2w}, x_v, x_{2g} \geq 0 \quad (4)$$

Note that a feasible solution is to have an initial value of 32 for each of the  $x_i$  and an initial value of 0 for each of the  $y_i$  - this yields an overall value of 0. To try to increase this value, we will try to increase the value of  $y_{pub}$ , as it has the highest coefficient in the maximization equation. We increase  $y_{pub}$  until its value hits 32, at which point  $x_{2w} = x_{2g} = 0$  (so the first and third constraints are tight) and our value goes up to 96. We can now rearrange Equation (3) to write  $y_{pub}$  as  $y_{pub} = 32 - x_{2g} - y_{mar}$  and plug this into Equation (1) to get

$$x_{2w} = 32 - y_{ws} - y_{man} - (32 - x_{2g} - y_{mar}) \quad (5)$$

$$= x_{2g} + y_{mar} - y_{ws} - y_{man}. \quad (6)$$

Note that our "active" variables are now  $y_{ws}$ ,  $y_{man}$ ,  $y_{mar}$ , and  $x_{2g}$ , and our "slack" variables are  $y_{pub}$ ,  $x_v$ , and  $x_{2w}$  and so our maximization equation becomes

$$y_{ws} + 2y_{mar} + 2y_{man} + 3(32 - x_{2g} - y_{mar}) = 96 + y_{ws} + 2y_{man} - y_{mar} - 3x_{2g}. \quad (7)$$

The natural choice here is to try to increase either  $y_{man}$  or  $y_{ws}$ . This is a problem, however, for Equation (6), as increasing either would force  $x_{2w}$  to become negative. Therefore, we need to further rewrite our equations, this time making  $y_{man}$  a slack variable and  $x_{2w}$  an active one. Therefore, we rearrange Equation (6) to have  $y_{man} = y_{mar} + x_{2g} - y_{ws} - x_{2w}$ . We can plug this into Equation (2) to get

$$x_v = 32 - (y_{mar} + x_{2g} - y_{ws} - x_{2w}) - y_{mar} \quad (8)$$

$$= 32 - 2y_{mar} - x_{2g} + y_{ws} + x_{2w}. \quad (9)$$

We can also plug it into Equation (7) to get

$$96 + y_{ws} + 2(y_{mar} + x_{2g} - y_{ws} - x_{2w}) - y_{mar} - 3x_{2g} = 96 - y_{ws} + y_{mar} - x_{2g} - 2x_{2w}. \quad (10)$$

Note that our LP problem has now become:

$$\text{Maximize } 96 - y_{ws} + y_{mar} - x_{2g} - 2x_{2w} \text{ subject to:} \quad (11)$$

$$y_{man} = y_{mar} + x_{2g} - y_{ws} - x_{2w} \quad (12)$$

$$x_v = 32 - 2y_{mar} - x_{2g} + y_{ws} + x_{2w} \quad (13)$$

$$y_{pub} = 32 - x_{2g} - y_{mar} \quad (14)$$

$$x_{2w}, x_v, x_{2g}, y_{ws}, y_{mar}, y_{man}, y_{pub} \geq 0 \quad (15)$$

and that we have a current value of 96,  $y_{pub} = x_v = 32$ , and  $y_{man} = y_{mar} = y_{ws} = x_{2w} = x_{2g} = 0$ . The natural thing to try to increase is  $y_{mar}$ , as it is the only variable in the maximization equation with a positive coefficient. If we increase until  $y_{mar} = 16$ , Equation (13) is now tight. Note that now we have  $y_{man} = y_{pub} = y_{mar} = 16$ , and everything else is 0. If we rewrite all our equations to make  $y_{man}$ ,  $y_{pub}$ , and  $y_{mar}$  the slack variables (I'll skip the algebra this time), we end up with

$$\text{Maximize } 112 - \frac{y_{ws}}{2} - \frac{3x_{2w}}{2} - \frac{x_v}{2} - \frac{3x_{2g}}{2} \text{ subject to:}$$

$$y_{man} = 16 + \frac{x_{2g}}{2} - \frac{x_v}{2} - \frac{x_{2w}}{2} - \frac{y_{ws}}{2}$$

$$y_{mar} = 16 + \frac{y_{ws}}{2} + \frac{x_{2w}}{2} - \frac{x_v}{2} - \frac{x_{2g}}{2}$$

$$y_{pub} = 16 + \frac{x_v}{2} - \frac{y_{ws}}{2} - \frac{x_{2w}}{2} - \frac{x_{2g}}{2}$$

$$x_{2w}, x_v, x_{2g}, y_{ws}, y_{mar}, y_{man}, y_{pub} \geq 0$$

Now we are done - all the coefficients in our maximization equation are negative and so we cannot improve it, which means that the optimal value of this LP (and thus the most the bartender can earn) is 112. This was achieved by making 0 whiskey sours, 16 martinis, 16 manhattans, and 16 pub specials.

Note that we did not use anywhere the amount that each bottle would cost. This is because we are only thinking about one night and therefore do not have to consider buying new bottles to replace the ones that were used. If we did, we would have to try to simultaneously maximize the revenue while minimizing the wholesale cost of the liquor, which would be a much more complicated problem.