CSCI2510 Approximation Algorithms
Assignment 5

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Simplex Exercise n.4

Let consider the LP:

$$
\begin{aligned}
\max \quad & x_1 \\
s.t. \quad & -x_1 + x_2 \leq 2 \\
& x_1 + x_2 \leq 8 \\
& -x_1 + x_2 \geq -4 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

The canonical form of this LP is:

$$
\begin{aligned}
\max \quad & x_1 \\
s.t. \quad & -x_1 + x_2 + w_3 = 2 \\
& x_1 + x_2 + w_4 = 8 \\
& x_1 - x_2 + w_5 = 4 \\
& x_1, x_2, w_3, w_4, w_5 \geq 0
\end{aligned}
$$

To solve the using the simplex algorithm, we need an initial feasible solution. Let us consider $x_1 = x_2 = 0$; it is a feasible solution, and we can rewrite our LP as:

$$
\begin{aligned}
\max \quad & x_1 \\
s.t. \quad & w_3 = 2 + x_1 - x_2 \\
& w_4 = 8 - x_1 - x_2 \\
& w_5 = 4 - x_1 + x_2 \\
& x_1, x_2, \quad w_3, w_4, w_5 \geq 0
\end{aligned}
$$

To increase the value of the solution, we can only increase x_1. From the third constraint, we obtain $x_1 \leq 4$, and this is the minimum upper bound on x_1 using the constraints. Now we set $x_1 = 4$ and rewrite our LP:

$$\begin{array}{lrl} \max & & 4 + x_2 - w_5 \\ s.t. & w_3 = & 6 - w_5 \\ & w_4 = & 4 - 2x_2 + w_5 \\ & x_1 = & 4 + x_2 - w_5 \\ & x_1, x_2, & w_3, w_4, w_5 \geq 0 \end{array}$$

To increase the optimal solution we increase x_2 to 2, that is the maximum value for a feasible solution (obtained by the second constraint), and rewrite the LP as:

$$\begin{array}{lrl} \max & & 6 - w_4 - \frac{3}{2}w_5 \\ s.t. & w_3 = & 6 - w_5 \\ & x_2 = & 2 - \frac{1}{2}w_4 - \frac{1}{2}w_5 \\ & x_1 = & 6 - \frac{1}{2}w_4 - \frac{3}{2}w_5 \\ & x_1, x_2, & w_3, w_4, w_5 \geq 0 \end{array}$$

Now we cannot increase the value of the objective function, thus we have that the optimal solution to the initial LP is $x_1 = 6, x_2 = 2$ of value 6.

Graphically the convex polytope given by the constraints is shown in gray in Fig. 1.

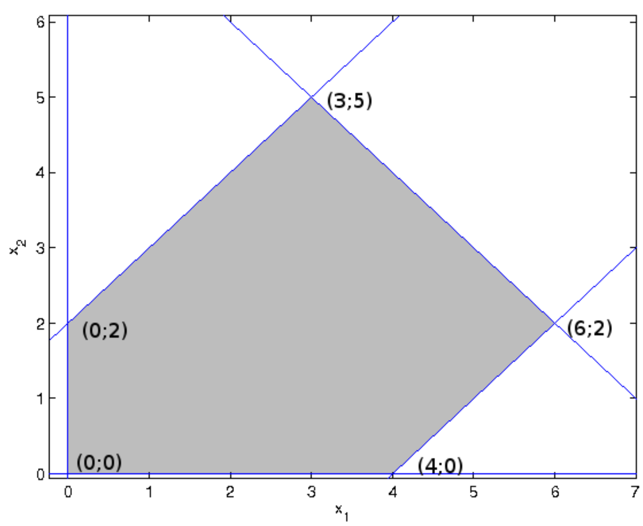

Figure 1: Convex polytope given by the constraints of the LP

The objective function depends only from x_1 and increase when x_1 increase (i.e., the gradient of the objective function is a vector with the same direction of the x_1 axis), thus the optimal solution corresponds to the vertex for which x_1 is maximal, i.e. $(x_1 = 6, x_2 = 2)$.

The algorithm start from the point $(x_1 = 0, x_2 = 0)$, it "moves" toward the third constraint reaching the solution $(x_1 = 4, x_2 = 0)$, then moves to the optimal solution $(x_1 = 6, x_2 = 2)$ satisfying the second constraint to the equality.

If now we change the objective function to $x_1 + cx_2$, we are changing the gradient of the objective function. First of all, let consider two extremal case:

- $c = 1$: in this case the optimal solutions are $(x_1 = 6, x_2 = 2)$ *and* $(x_1 = 3, x_2 = 5)$.
- $c = -1$: in this case the optimal solutions are $(x_1 = 6, x_2 = 2)$ *and* $(x_1 = 4, x_2 = 0)$.

It easy to see that the solution $(x_1 = 6, x_2 = 2)$ is the only optimal solution if $-1 < c < 1$ (like in the initial LP, with $c = 0$); thus $(x_1 = 6, x_2 = 2)$ is an optimal solution for every $-1 \leq c \leq 1$.

The shadow price of a constraint is the change in the objective function obtained increasing the RHS (of the constraint) by a unit. If we increase the RHS of the third constraint (in canonical form) to 5 and consider the initial objective function, we have that the optimal solution is still given by the vertex at the intersection of the second and the third constraint, and that this is the point $(x_1 = 6.5, x_2 = 1.5)$. Thus the shadow price of the third constraint is $1/2$.