CS 251 - Problem 14.4

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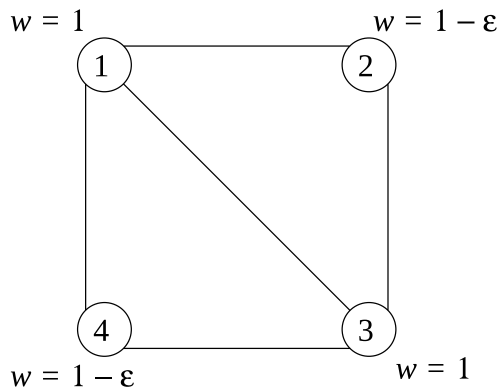
Consider the following graph, with weights assigned to its vertices:



The optimal vertex cover for this graph is clearly $\{v_1, v_3\}$, and therefore has weight 2. In order to show that this is a tight example for our half-integrality-based algorithm, we must first find the solution that the algorithm would find. We can start by writing the problem as an LP-relaxation of the integer program:

$$\begin{aligned} \text{minimize } & x_1 + x_3 + (1-\varepsilon)\cdot(x_2 + x_4) \\ \text{subject to } & x_1 + x_2 \geq 1 \\ & x_1 + x_3 \geq 1 \\ & x_1 + x_4 \geq 1 \\ & x_2 + x_3 \geq 1 \\ & x_3 + x_4 \geq 1 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

The optimal solution for this LP is $x_1 = x_2 = x_3 = x_4 = 1/2$, which results in a value of $2 - \varepsilon$ for the objective function. In fact, this is the only combination of weights that produces that value for the objective function. Therefore, this solution is extreme, and the algorithm will pick it.

The vertex cover that the algorithm outputs will contain all four vertices (because it picks all vertices that are set to one half or one), for a total weight of $4 - 2\varepsilon$. This is twice the weight of the optimal solution, and so we have a tight example.