# Homework 6

## Problem 14.6

*Give a counterexample to the following claim. A set cover instance in which each element is in exactly $f$ sets has a $(1/f)$-integral optimal fractional solution (i.e., in which each set is picked an integral multiple of $1/f$).*

Consider the following set cover instance:

$$U = \{a, b, c, d\}$$

$$S = \{\{a, b\}, \{a, d\}, \{a, b, c\}, \{b, c, d\}, \{c, d\}\}$$
$$\forall s \in S, c(s) = 1$$
$$f = 3$$

The LP relaxation of this instance is given by:

$$\begin{array}{ll} min & \sum_{s \in S} c(s) x_s \\ s.t. & \sum_{s \ni e} x_s \geq 1 \quad e \in U \\ & x \geq 0 \end{array}$$

Where $x_s = 1$ for each constraint representing an element $e$ such that $e \in s$. This is equivalent to:

$$\left( \begin{array}{c|ccccc} & S_1 & S_2 & S_3 & S_4 & S_5 \\ \hline a & 1 & 1 & 1 & & \\ b & 1 & & 1 & 1 & \\ c & & & 1 & 1 & 1 \\ d & & 1 & & 1 & 1 \end{array} \right) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \geq \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

The simplex algorithm gives an optimal solution as $x = \left(0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0\right)$. However, because $f = 3$ and $\frac{1}{2}$ is not an integral multiple of $\frac{1}{3}$, this set cover instance constradicts the original claim.